Exploring Elementary Mathematics

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A Note to the User

As you explore this book you will notice that it is not a traditional text book, and your first indication should have been that this page is titled "A Note to the **User**" rather than "A Note to the **Reader**". This book is not meant to be passively read, but rather used as a tool to engage in the extremely rich and interesting aspects of elementary mathematics. Let's begin by discussing the term *elementary mathematics*. First and foremost, the words elementary and easy are NOT synonyms. Elementary mathematics is the foundation that all of mathematics is built on, and you know it as the mathematics curriculum covered in kindergarten through eighth grade. Because it is taught at such an early age many people brush it off as basic, missing out on the rich and interesting mathematics hidden within. This book gets you started on a journey of discovery that I hope you continue throughout your career as a teacher. (Even if you aren't going to be a teacher, there's still some really cool math to explore just for fun!)

Although I said you should not just passively read this book, I do still want you to read it. To this end, I chose to use a style much less formal than a traditional text book. In fact, I wrote it as if I were talking to you. Consequently, I will use the pronouns I, we, and you. (The use of I and you will drive the adherents to formal textbook writing crazy!) Although you may not know me, those that do will recognize my personality in this book, and so for that I apologize ahead of time. What can I say, I'm a bit of a nerd and proud of it!

Finally, if I want you to use this book productively, I should probably tell you how it is structured. First and foremost you need to know that this textbook is not complete. It is your job to complete it as you proceed through the text. You will have to decide the mechanics of how you complete the book. Will it be handwritten notes with labeling references to the book? Will it be electronically within the text itself? That choice is up to you. On my end, I have labeled things in a way that I hope will make it clear what is expected from a user of this book.

There are examples and solutions throughout the book, but of course there is always more than one way to arrive at an answer. The ability to recognize and understand multiple methods is of great value to a teacher because it is quite impressive how creative kids can be with their solutions. To develop this ability, whenever you see an example whose solution is labeled **Possible Solution**, then you should take the time to think about and write down alternate solutions.

Throughout the book you will see the label **Question**. The questions posed under this label are not answered in the text. It is your responsibility to record the answers to those questions as well as the reasoning that justifies the given answer.

Finally, you will see the label **Property**. Statements under this label are either facts that we have discovered or shortcuts we have found along our path of discovery that we think will be useful to have at our disposal moving forward. In our exploration, we will discover possible properties by looking at examples. However, that does not necessarily mean the statement is always valid. Maybe it won't work with a different example. Consequently, before applying this property moving forward, we must be sure it is valid. Therefore for anything labeled as a property you should record an argument justifying the validity of the said property in general. In other words, your argument should explain why the statement is true no matter what example you are using.

Well, I think that's all I have to say about that. Let's go do some math!

Contents

1 Algebraic Thinking **5**

1.1 Patterns 5

1.2 Exercises 9

2 Fenland **13**

2.1 Introduction 13

2.2 Fenland Money 15

2.3 Numerals 17

2.4 Number Lines 21

2.5 Addition 23

2.6 Subtraction 26

2.7 Multiplication (Part I) 31

2.8 Properties 33

2.9 Multiplication (Part II) 37

2.10 Division 43

2.11 Exercises 48

3 Back to Base Ten **54**

3.1 Re-exploring the System We Know 54

3.2 Exercises 56

4 Integers **59**

4.1 Definition of Negative Numbers 59

4.2 Addition of Integers 61

4.3 Subtraction of Integers 63

4.4 An Alternate Model for Integers 66

4.5 An Integer Interlude 70

4.6 Multiplication and Division of Integers . . . 73
4.7 Exercises . . . 76

5 Exponents (Part I) 79
5.1 Definition of Exponents . . . 79
5.2 Properties of Exponents . . . 80
5.3 Exponents Looking Forward . . . 82
5.4 Exercises . . . 84

6 Number Theory 85
6.1 Fundamental Theorem of Arithmetic . . . 85
6.2 Finding Factors Using Prime Factorization . . . 87
6.3 Finding Multiples Using Prime Factorization . . . 90
6.4 GCF and LCM Using Prime Factorization . . . 91
6.5 Exercises . . . 97

7 Geometry (Part I) 99
7.1 Perimeter and Area . . . 99
7.2 Formulas for Area . . . 102
7.3 Surface Area . . . 104
7.4 Volume . . . 106
7.5 Exercise . . . 110

8 Fractions 115
8.1 Building on Previous Knowledge . . . 115
8.2 Definition of a Fraction . . . 116
8.3 Fractions on the Number Line . . . 121
8.4 Equivalent Fractions . . . 124
8.5 Applications of Equivalent Fractions . . . 127

8.6 Multiplication . . . 130
8.7 Division . . . 133
8.8 Numerators and Denominators That Are Not Whole Numbers . . . 137
8.9 Exercises (Part I) . . . 140
8.10 Exercises (Part II) . . . 145

9 Exponents (Part II) 152

9.1 Extending the Basics . . . 152
9.2 Fractional Exponents . . . 153
9.3 Exercises . . . 156

10 Decimals 158

10.1 Place Value Continued . . . 158
10.2 Operating with Decimals . . . 161
10.3 Powers of Ten and Decimals . . . 167
10.4 Terminating or Not? . . . 171
10.5 Exercises . . . 176

11 Rational and Irrational Numbers 179

11.1 Rational Numbers Revisited . . . 179
11.2 Irrational Numbers . . . 181
11.3 Exercises . . . 183

12 Ratios and Proportional Reasoning 185

12.1 Algebra Problems Before Algebra . . . 185
12.2 Ratios vs. Fractions . . . 185
12.3 Proportional or Not? . . . 191
12.4 Proportional Reasoning Before Algebra . . . 191
12.5 Exercises . . . 196

13 Geometry (Part II) **201**

13.1 Pythagorean Theorem and Beyond 201

13.2 Angles 204

13.3 Exercises 205

1 Algebraic Thinking

An understanding of algebraic concepts is extremely important to a student's success in mathematics. Finding ways to ground students, even at an early age, in algebraic thinking will promote their success later on. In this chapter we will be looking at a few examples that will develop a student's algebraic thinking. By altering the questions you ask and how far you progress, activities of this nature can be used at any grade level. In other words, even if you are teaching first grade you can help students get ready for algebra.

1.1 Patterns

Consider the figures below.

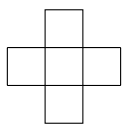


Figure #1

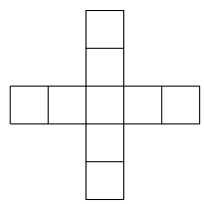


Figure #2

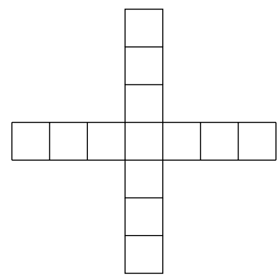


Figure #3

Question 1.1.1. What does figure #4 look like?

Example 1.1.2. How many squares are in figure #4?

Possible Solution. If the growth follows the pattern of adding one block to each leg, then Figure #4 has 13 squares. □

Now, here's where some algebraic thinking begins. Did you actually count all of the squares to get 13, or did you have a quicker way to figure out that there were 13 squares?

Example 1.1.3. Without actually counting each square, determine how many squares figure #5 will have.

Possible Solution. There are lots of ways to figure this one out, but here's one way. We can think of this figure as a square in the center and four arms. Notice that figure #1 has 1 square in each arm, figure #2 has 2 squares in each arm, etc. Using this structure we found in the figures, we can now determine the arithmetic needed to find the number of blocks in each figure.

Figure #	Work	Number of Squares
1	$4 \times 1 + 1$	5
2	$4 \times 2 + 1$	9
3	$4 \times 3 + 1$	13
4	$4 \times 4 + 1$	17

Looking at the pattern in the table, we can see that figure #5 will have 5 squares in each arm, so we have four arms with 5 in each arm plus the square in the middle. Using this information we see that figure #5 will have $4 \times 5 + 1$ or 21 squares. □

Example 1.1.4. Determine how many squares figure #35 will have?

Possible Solution. Figure #35 will have 4 arms, with 35 squares in each arm. Plus it will also have one square in the middle. Therefore to determine the number of squares we need to compute $4 \times 35 + 1$, so there are 141 squares in figure #35. □

As stated above, there is more than one way to count the squares. Suppose a student saw the figures growing. It starts with figure #1, and at each step four squares are added on the end of each arm. Now figure #2 has 1 set of 4 squares added. Figure #3 has 2 sets of 4 squares added. Figure #4 has 3 sets of 4 squares added. Continuing we see figure #5 has 4 sets, figure #6 has 5 sets, etc. Therefore we can conclude that figure #35 will have 34 sets. Using this information, we see that we could have arrived at 141 squares by computing the base figure plus 34 sets of 4. In other words, $5 + 4 \times 34$.

Question 1.1.5. A student arrived at 141 squares in figure #35 by doing the following arithmetic, $3 \times 35 + 36$. What structure do you think the student sees in the figures that led to this arithmetic?

Let's now return to our first method of counting, namely four arms and a square in the center. For figure #35 we had the expression $4 \times 35 + 1$. If we go out to figure #100, what will change in the expression? Will the 4 change? Will the 35 change? Will the 1 change? In answering these questions, we see that the corresponding expression for figure #100 would be $4 \times 100 + 1$. Consequently, there are 401 squares in figure #100.

We can easily see that as we change figures, the 4 and the 1 remain the same in the expression because the 4 represents the number of arms and the 1 represents the square in the middle. Therefore there is only one number that varies in the expressions. We can now see how to find an algebraic expression that represents the number of squares in figure $\#n$. There will still be four arms and 1 square in the middle, the only change is the number of squares in an arm. For figure #35 there were 35 squares in an arm, for figure #100 there were 100 squares in an arm. Thus we can see that in figure $\#n$ there will be n squares in an arm. Therefore the expression $4n + 1$ tells us how many squares are in figure $\#n$.

In reviewing what we did above, we essentially looked for a pattern in the many expressions we wrote down. In addition, we were able to correlate each number in the expression to what it represented in the figure. If students are able to do these two things successfully, then the jump from concrete numbers to abstract variables will be an easier transition. One of the struggles students have in algebra is understanding the different roles a variable can play. When we found the formula for the nth figure above, the variable played the role of "any whole number". In other words, n did not stand for a specific number that we were solving for, but rather it represented a number that can vary. The corresponding expression represented the number of squares as the figure number varied.

Let's look at a seemingly different problem but that can be attacked in a similar way. For Examples 1.1.6 through 1.1.8, consider the sequence of numbers below.

$$3, 10, 17, 24, 31, 38, \ldots$$

Example 1.1.6. What is the next number in the sequence?

Possible Solution. Looking at the pattern we see that to get from one term to the next we add 7. Therefore the next number will be 45. □

Example 1.1.7. What is the 35th number in the set?

Possible Solution. Let's use the idea with the squares above where we look at our work to see a pattern. To get from 3 to 10 we added 7. To get from 10 to 17 we added 7, which means to get from 3 to 17 we must add 7 twice. Similarly to get from 3 to 24 we must add 7 three times. We can summarize these results in the table below.

n	work	nth number in the set
1	—	3
2	$3 + 7$	10
3	$3 + 2 \times 7$	17
4	$3 + 3 \times 7$	24
5	$3 + 4 \times 7$	31
6	$3 + 5 \times 7$	38

Now we can see a pattern. Notice that the 3 doesn't change and the 7 doesn't change. The only thing that changes is how many 7's we are adding. Notice that for the 4th term we added 3 sevens, in the 5th term we added 4 sevens, in the 6th term we added 5 sevens, etc. Therefore we can see that in the 35th term we will add 34 sevens. Therefore we can get the 35th term by computing $3 + 34 \times 7$. So 238 is the 35th number in this set. □

Example 1.1.8. Will the number 576 appear in the sequence?

Possible Solution. From our above work we know that numbers in this set come from an expression like $3 + \_\_ \times 7$, where we can fill in the blank with some whole number. In other words we are trying to finding a whole number n, so that $3 + n \times 7$ is equal to 576. Therefore we need to solve the equation $3 + n \times 7 = 576$.

$$\begin{aligned} 3 + n \times 7 &= 576 \\ n \times 7 &= 573 \\ n &\approx 81.86 \end{aligned}$$

Since n does not turn out to be a whole number, then 576 is not in the set. □

This section gave just a very brief introduction to what algebraic thinking might look like. The more students are introduced to pattern recognition and description of patterns in general the more prepared they will be to deal with the notion of variable as a ranging quantity. Notice in the last example, the variable started out as "any number" because the formula represented the value of a term where n represented any term number. However, once we were asked to determine if the specific number, 576, was in the sequence we then shifted to determine a single value for the variable. In other words, we then solved to find a specific value for n. Algebra is full of variables playing different roles, which often leads to confusion. In particular, students tend to think of variables as what you solve for, so if they are exposed to some of these algebraic thinking activities throughout their elementary career they will have the ground work laid to see variables playing a different role.

1.2 Exercises

1. Consider the pattern of block figures shown below.

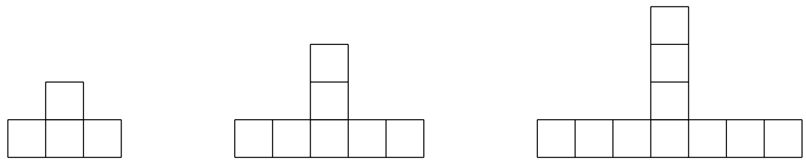


Figure 1 Figure 2 Figure 3

   (a) Describe a structure you see in the figures, and use that structure to determine the number of blocks in Figure 3.

   (b) Using the structure you described above to produce a corresponding arithmetic expression that will give the number of blocks in Figure 68?

   (c) Using the structure you described above to produce a corresponding algebraic expression that will give the number of blocks in Figure n?

   (d) Use the structure you described above to determine if there is a figure in this pattern that has 316 blocks. Your explanation should rely solely on the blocks themselves.

   (e) Show the algebraic solution that corresponds to your contextual solution in Problem 1d.

2. Consider the pattern of block figures below.

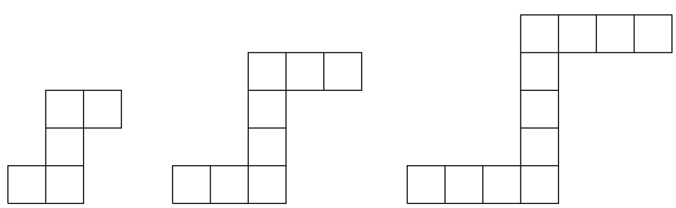


Figure 1 Figure 2 Figure 3

   (a) Max says that there are 11 blocks in Figure 3, and he used the arithmetic expression $3 + 2 \times 4$ to arrive at his answer. If Max continues the same thinking, what arithmetic expression do you think he will use to find the number of blocks in Figure 5? Please explain how you arrived at your answer.

   (b) Macy is looking at the figures and she says that she sees the figures as one long piece in the middle and two short arms off the side. So for example in figure #3 Macy sees a long piece in the middle with 5 blocks and two arms that each have 3 blocks.

   Complete the table below using the structure Macy sees in the figures.

Figure #	Work	Number of Squares
1		5
2		8
3		11
4		14

(c) Use Max's structure to produce an algebraic expression that will give the number of blocks in Figure n?

(d) Use Macy's structure to produce an algebraic expression that will give the number of blocks in Figure n?

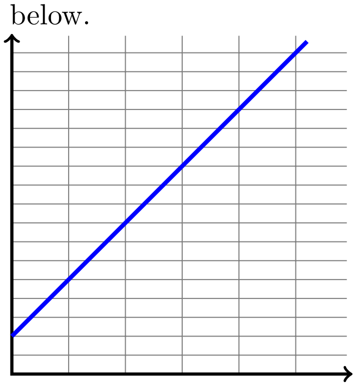
3. Another student, Marta, is working on the same pattern of blocks in Exercise 2, and her formula for Figure n is $3n + 2$.

   (a) What structure do you think Marta is seeing in the figures?

   (b) Marta's teacher asked the class a question about the blocks. Marta thinks about her teacher's question, and then writes down the inequality $3n + 2 \geq 100$. What question do you think the teacher asked about the blocks?

   (c) In solving the inequality $3n + 2 \geq 100$, Marta's second step is $3n \geq 98$. Use the context of the blocks to explain why the second inequality follows from the first.

   (d) Marta wrote down her formula, $T = 3n + 2$, and then graphed it. Her graph is shown below.



   i. What should the labels on the x- and y-axes be?

   ii. What does the y-intercept of the line represent in the context of the squares?

   iii. What does the slope of the line represent in the context of the squares?

4. Consider the pattern of block figures below. Please justify all responses to the following questions.

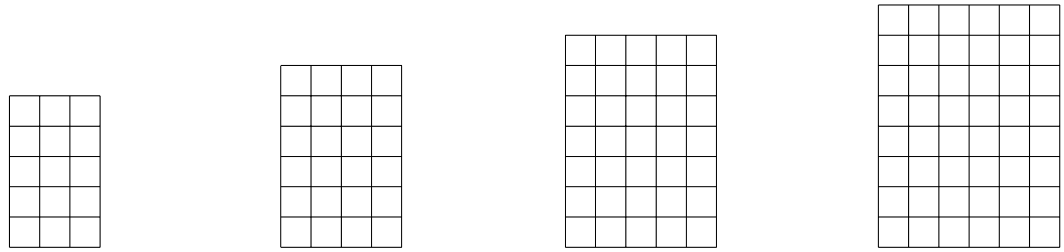


Figure 3 Figure 4 Figure 5 Figure 6

(a) How many blocks will be in Figure 20?

(b) How many blocks will be in Figure n?

(c) Is there a figure number for which there will be 63 blocks?

(d) Is there a figure number for which there will be 100 blocks?

5. Consider the pattern of block figures below.

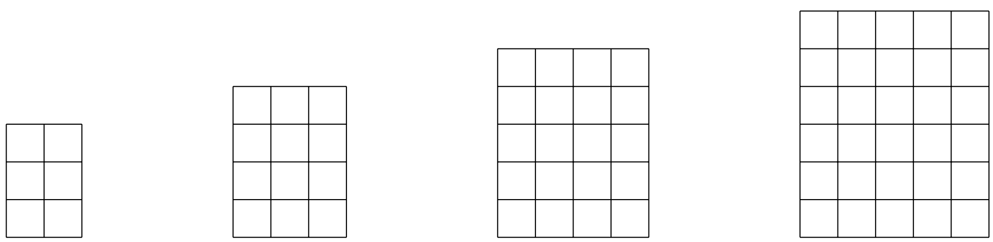


Figure 1 Figure 2 Figure 3 Figure 4

Jazlyn says her formula for the number of blocks in Figure n is $(n+1)(n+2)$. Jacarri says his formula for the number of blocks in Figure n is $n^2 + 3n + 2$. Jazlyn used algebra to see that her formula and Jacarri's formula are equivalent, however she does not see how Jacarri could have gotten that formula from the pictures. Where does each part of Jacarri's formula appear in the figures?

6. For each of the algebraic expressions below, create a pattern of *rectangular* figures for which the expression would represent the number of squares in Figure n.

(a) $n^2 + 5n + 6$ (b) $n^2 + 5n + 4$ (c) $n^2 + 8n + 12$

7. For each of the algebraic expressions below it is not possible* to create a pattern of *rectangular* figures for which the expression would represent the number of squares in Figure n. Instead, create patterns of figures that are as close as possible to rectangles. Indicate how many squares too much or too little you are from creating rectangles.

(*Note: We are assuming only full squares can be used. If we remove that criteria, then it could be possible.)

(a) $n^2 + 2n + 3$ (b) $n^2 + 8n + 5$ (c) $n^2 + 6n + 7$

8. In Exercises 6 and 7 we explored quadratic expressions that either could or could not be represented by a rectangular array of blocks. What is the connection between those two exercises and factoring quadratic expressions?

2 Fenland

2.1 Introduction

In our number system we only have ten digits (namely 0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and we build our number system out of those ten digits. In Fenland (an imaginary country), they only use five digits (namely 0, 1, 2, 3, 4). We need to learn the fen number system. When a Fenlander sees this many stars: ⋆ ⋆ ⋆ ⋆ ⋆ , they would say there are **fen** stars. Let's look at several examples to begin to understand the number names.

Example 2.1.1. How many stars?

⋆ ⋆ ⋆ ⋆ ⋆ ⋆

Answer. There are fen one stars.

Example 2.1.2. How many stars?

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

Answer. There are fen three stars.

Example 2.1.3. How many stars?

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

Answer. There are two fen stars.

Example 2.1.4. How many stars?

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

Answer. There are two fen four stars.

With these examples, you should be able to count stars until we get to this many:

⋆ ⋆ ⋆ ⋆ ⋆
⋆ ⋆ ⋆ ⋆ ⋆
⋆ ⋆ ⋆ ⋆ ⋆
⋆ ⋆ ⋆ ⋆ ⋆
⋆ ⋆ ⋆ ⋆ ⋆

In the above there are fen groups and in each group there are fen stars. A Fenlandar would say there are **one fefen** stars.

Example 2.1.5. How many stars?

⋆ ⋆ ⋆ ⋆ ⋆   ⋆ ⋆ ⋆ ⋆ ⋆   ⋆ ⋆ ⋆ ⋆ ⋆
⋆ ⋆ ⋆ ⋆ ⋆   ⋆ ⋆ ⋆ ⋆ ⋆   ⋆ ⋆ ⋆ ⋆ ⋆
⋆ ⋆ ⋆ ⋆ ⋆   ⋆ ⋆ ⋆ ⋆ ⋆   ⋆ ⋆ ⋆ ⋆ ⋆
⋆ ⋆ ⋆ ⋆ ⋆   ⋆ ⋆ ⋆ ⋆ ⋆   ⋆ ⋆ ⋆ ⋆
⋆ ⋆ ⋆ ⋆ ⋆   ⋆ ⋆ ⋆ ⋆ ⋆

Answer. There are two fefen three fen four stars.

Now, the next time we need a new name is when we have fen groups and in each group there are one fefen stars.

⋆ ⋆
⋆ ⋆
⋆ ⋆
⋆ ⋆
⋆ ⋆

A fenlander would say there are **one fefefen** stars.

Example 2.1.6. How many stars?

⋆ ⋆
⋆ ⋆
⋆ ⋆
⋆ ⋆
⋆ ⋆

⋆ ⋆
⋆ ⋆
⋆ ⋆
⋆ ⋆
⋆ ⋆

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆
⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆
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⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆
⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

Answer. There are two fefefen three fefen four fen one stars.

Reconsidering the names above, we basically learned three new words; fen, fefen, and fefefen. We used the words one fefen to mean fen fens. (This should be reminiscent of hundred in our number system, because we use the word one hundred to mean ten tens.) Similarly we used the words one fefefen to mean fen fefens. (This should be reminiscent of thousand in our number system, because we use the word one thousand to mean ten hundreds.) Considering our number system, we should be able to determine when a new name is needed in the Fenland system. For example, if we have ten thousands in our number system, do we have a special name for that? No. We just call that number ten thousand. So when do we need a new name in our system? Not until we get to a million. Similarly in the Fen system, if we have fen fefefens, we just call that number fen fefefen. Similarly the next step up would be one fefen fefefen. (This would be reminiscent of one hundred thousand.) Finally we get to the need for a new name in Fenland. If we have fen groups of one fefen fefefen, we would call that number one mefefen. We should now have enough understanding to know when we need a new word, the only thing we would need to know is what the next new word actually is. What do you think Fenlanders' next new word is?

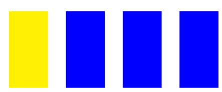
2.2 Fenland Money

In Fenland there are four denominations of paper money. They have a blue bill, a yellow bill, a green bill, and a pink bill.

- The lowest denomination is blue.
- One yellow has the same value as fen blues.
- One green has the same value as fen yellows.
- One pink has the same value as fen greens.

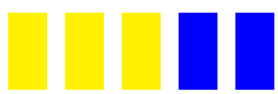
Let's consider how we talk about money in the United States. Suppose you had two twenties, a five, and three ones in your wallet. If someone asked you how much money you have in your wallet, you most likely would not say, "I have two twenties, one five, and three ones." More likely, you would say, "I have forty eight dollars." Just as we do in the United States, Fenlanders call their lowest denomination bill a dollar. So for example, if a Fenlander had three blue bills, they would say they have three dollars.

Example 2.2.1. Shaya has the following bills in her wallet. How much money does she have?



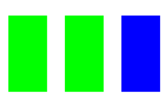
Answer. Fen three dollars.

Example 2.2.2. Angus has the following bills in his wallet. How much money does he have?



Answer. Three fen two dollars.

Example 2.2.3. Jaxton has the following bills in his wallet. How much money does he have?



Answer. Two fefen one dollars.

Example 2.2.4. Bindi has the following bills in her wallet. How much money does she have?

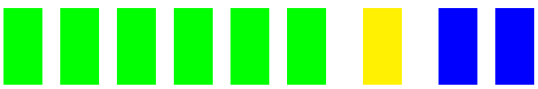
Answer. One fefefen two fefen fen one dollars.

Example 2.2.5. Joe has 2 greens, 3 yellows, and 4 blues. Jane has 3 greens, 2 yellows, and 3 blues. They combine their money and take it to the bank. They give the teller their money and ask him to exchange the money for as few bills as possible. What bills does the teller give to Joe and Jane?

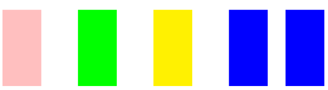
Possible Solution. Combining their money, here's what Joe and Jane give to the teller:

The teller exchanges fen blues for a yellow, so now the teller has:

Next the teller exchanges fen yellows for a green, so the teller has:



Finally the teller exchanges fen greens for a pink, so the teller ends up with:



The teller gives 1 pink, 1 green, 1 yellow, and 2 blues to Joe and Jane. □

Question 2.2.6. How much money do Joe and Jane have all together?

Below are a few more money problems for you to work on. Only the answers have been provided so be sure you play with the money and see how to get the answer purely by manipulating the money.

Example 2.2.7. Misha has one pink, three greens, two yellows, and one blue saved up. Her best friend's favorite number is four, so she wants to give her best friend four greens, four yellows, and four blues for her birthday. Misha goes to the bank to exchange her bills so that she can give her friend the birthday present. What bills does Misha have left after giving her friend the money?

Answer. Three greens, two yellows, and two blues.

Question 2.2.8. How much money does Misha have left?

Example 2.2.9. Ahmed gets two yellows and three blues every week for allowance. If he saves his allowance for four weeks and then goes to the bank to exchange his bills so that he has as few bills as possible, what bills will he end up with?

Answer. Two greens and two blues.

Question 2.2.10. How much money does Ahmed have saved up?

Example 2.2.11. Joe has four fefefen one fefen four fen three dollars. He wants to give each of his four friends the exact same pile of money. He is at the bank to make any exchanges he needs. What bills will Joe give to each friend?

Answer. One pink, two yellows, and two blues.

Question 2.2.12. How much money does each friend get?

2.3 Numerals

Now that we know the names of the numbers in Fenland, we need to discuss a numeration system so that we don't have to keep writing the name of the numbers in words. We know that in Fenland, they only have the digits 0,1,2,3, and 4, so we must build our numeration system out of those five digits. In our numeration system we need to keep track of how many ones, how many fens, how many fefens, etc.

For example, consider the group of stars below. We can see that there are two groups that have fen stars in them, and then three stars at the end. Therefore there are two fen three stars. To record this in numeral form, we note that we have 2 fens and 3 ones. Therefore a Fenlander would write the numeral 23 to represent the number two fen three.

$$\star\ \star\ \star\ \star\ \star \quad \star\ \star\ \star\ \star\ \star \quad \star\ \star\ \star$$

As long as we know our place values, we can represent any number with a numeral. Let's figure out what our place values are. We start with ones. When we have fen ones, we say we have fen. When we have fen fens, we say we have one fefen, so that is our next place value.

Example 2.3.1. How many stars are there? (Write your answer in numeral form.)

$$\begin{array}{lll}
\star\ \star\ \star\ \star\ \star & \star\ \star\ \star\ \star\ \star & \star\ \star\ \star\ \star\ \star \\
\star\ \star\ \star\ \star\ \star & \star\ \star\ \star\ \star\ \star & \star\ \star\ \star\ \star\ \star \\
\star\ \star\ \star\ \star\ \star & \star\ \star\ \star\ \star\ \star & \star\ \star\ \star\ \star\ \star \\
\star\ \star\ \star\ \star\ \star & \star\ \star\ \star\ \star\ \star & \star\ \star\ \star\ \star \\
\star\ \star\ \star\ \star\ \star & \star\ \star\ \star\ \star\ \star &
\end{array}$$

Answer. We see that we have 2 fefens, 3 fens and 4 ones. Therefore there are 234 stars. Remember however, that when we see 234, we say two fefen three fen four because we are in Fenland. □

Because we may be going back and forth between our number system and the Fenland number system, we want to try to avoid confusion. When we are working with numerals in the Fenland system we will put an f as a subscript to indicate we are working in Fenland. For example, we want to write down the numeral that represents two fefen three fen four (as in the above example), from now on we will write 234_f. The only time we may not use the subscript is if we have a single digit. A single digit means the same thing in either system, and so out of pure laziness I will often forego the use of a subscript on a single digit. For example 4 in the Fenland system is the same as 4 in the U.S. system, so I will usually not write 4_f. (If you decide you want to always write the subscript however you are welcome to do so.)

Example 2.3.2. Write the numeral that represents the number three fefen four fen two.

Answer. We have 3 fefens, 4 fens, and 2 ones, so the numeral is 342_f. ☐

Example 2.3.3. Write the numeral that represents the number two fefefen three fen four.

Answer. We have 2 fefefens, no fefens, three fens, and four ones, so the numeral is $2,034_f$. ☐

Example 2.3.4. Write the name of the number represented by $3,412_f$.

Answer. We have 3 fefefens, 4 fefens, 1 fen, and 2 ones, so the number is three fefefen four fefen fen two. ☐

Now that we have the idea of our numeration system, we need only remember the names of each place value. See the chart below.

tefefens				befefens				mefefens				fefefens		fefens	fens	ones
____	,	____	____	____	,	____	____	____	,	____	____	____	,	____	____	____

Question 2.3.5. What place value do each of the unlabeled blanks in the chart above represent?

Example 2.3.6. Write the numeral that represents three fen two fefefen four fefen two fen three.

Answer. The first thing we notice is how many fefefens we have, namely three fen two fefefens. Splitting up the three fen two into three fen and two, we see that three fen two fefefens is really three fen fefefens and two fefefens. To make the emphasis on place value more clear, we have 3 fen fefefens and 2 fefefens. Therefore we will need a 3 in the fen fefefens place and a 2 in the fefefens place. In addition, we have 4 fefens, 2 fens, and 3 ones. Therefore the numeral is $32,423_f$. ☐

Example 2.3.7. Write the number $234,012_f$ in words.

Answer. Two fefen three fen four fefefen fen two. ☐

Question 2.3.8. Look back at Example 2.2.5. How much money did Joe have originally? How much money did Jane have originally? How much money do they have all together?
(Write your answers as numerals.)

Now that we have a naming system and a numeral system in Fenland, we want to take some time to develop some flexibility with numbers. Let's consider numbers in the U.S. system for a minute. When we imagine the number 8 for example, we may actually view this number in several different ways. For example, we may see 8 as 4 and 4. We may also see 8 as 5 and 3. This flexibility often allows us to perform computations in our head more easily. Let's consider one more example from our number system, say the number 78. We can think of this number as 7 tens and 8 ones or as 70 and 8. However, we could also think of it as 60 and 18, or 6 tens and 18 ones. If we wanted to really go crazy, we could think of it as 4 tens and 38 ones.

To develop this flexibility, it is sometimes helpful to have a way to visually manipulate a number. Two tools often used in K-6 classrooms are place value cards and base ten blocks. The two videos below show these tools being used. We will also use these tools as we work in Fenland (with the necessary adjustments for the Fenland number system).

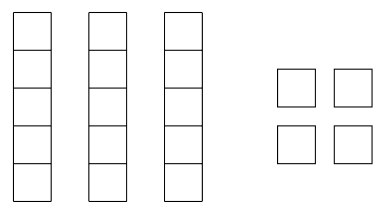
Place Value Cards Video

Base Ten Blocks Video

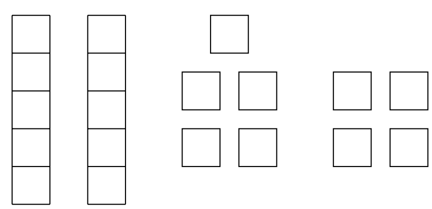
Question 2.3.9. What adjustments must be made to the tools introduced in the videos so that they can be used for a Fenland number system?

A third tool that helps with visualization is the number line. This has an even farther reaching use than the tools above because if students are comfortable working with whole numbers on a number line, then when they start working with fractions they can use their facility with number lines to develop a stronger understanding of fractions.

Let's shift back into the Fenland system and think about the number three fen four, which is represented by the numeral 34_f. That means we have 3 fens and 4 ones, which we have represented using base fen blocks below.

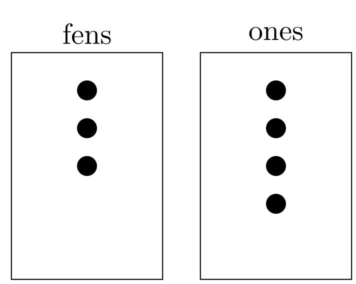


However, we could break up one of those rods into fen singles to get a new base fen representation of the same number. Therefore the base fen tiles below also represent the number 34_f.

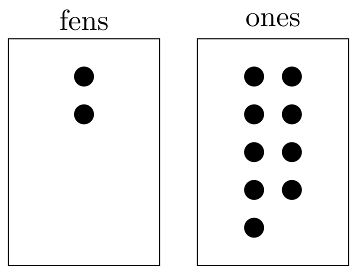


When we arrange the tiles as above, we are no longer thinking about the number three fen four as 3 fens and 4 ones, but rather as 2 fens and 14_f ones. Written in numeral form, that means we are thinking of the number 34_f as 20_f and 14_f.

Let's revisit this same process using place value cards. We first represent three fen four in the standard way by recognizing the number as 3 fens and 4 ones. This is shown below.

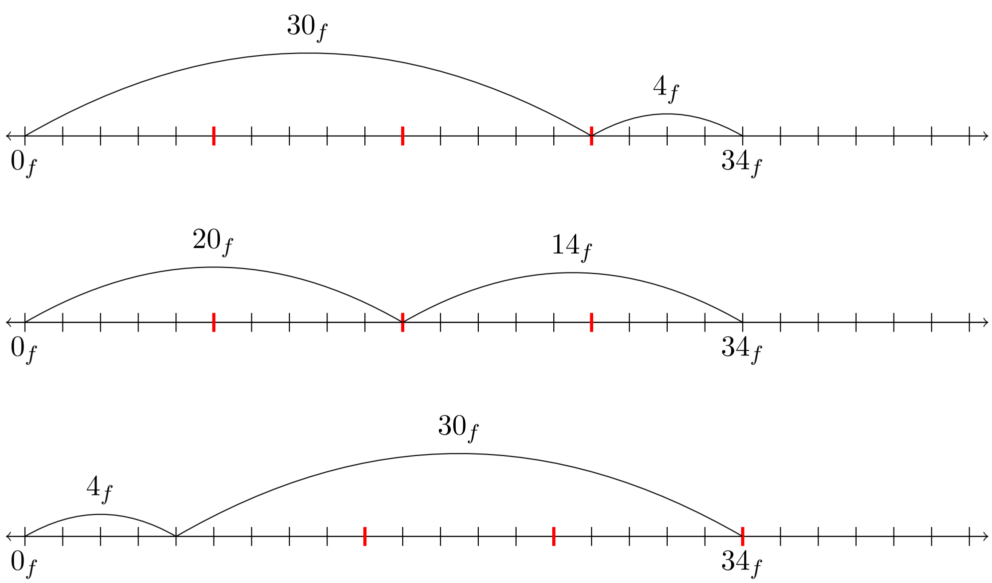


Because one fen has the same value as fen ones, then we can take one of our fen markers and exchange it for fen markers in the ones card. This is shown below.

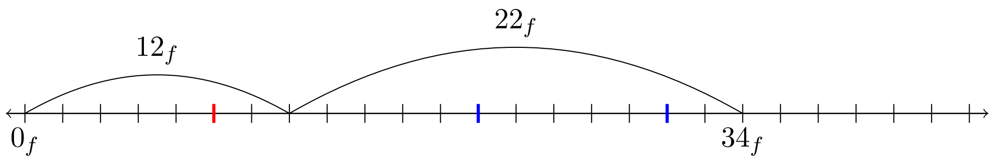


The above place value cards still represent the number three fen four, however rather than thinking of this number as 3 fens and 4 ones, we are thinking of this number as 2 fens and 14_f ones.

Let's now consider what this process would look like on a number line. We mark a distance of 34_f from 0 to place the number three fen four. The first representation below is thinking of 34_f as 30_f and 4_f. The second representation is thinking of 34_f as 20_f and 14_f. An added bonus to the number line is that we also have a way to think about order. For example, if a student is thinking of 34_f as 4_f and 30_f, then they might show it as in the third representation.



When we are using the base fen tiles or the place value cards we are limited in how we can break up our number. In particular, we must always break the number down into ones, fens, fefens, etc. However, with the number line we have much more flexibility. For example, on a number line we can show that we are thinking of 34 as 12 and 22, as shown below.



Example 2.3.10. Sadee said she has 214_f dollars in her wallet. Give three different sets of bills that she could have in her wallet.

Possible Solution. The most obvious possibility is that she has 2 greens, 1 yellow, and 4 blues.

Since 1 yellow has the same value as 10_f blues, then another possibility is that she has 2 greens and 14_f blues.

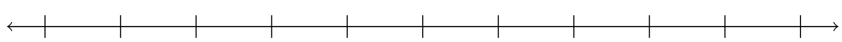
Since 1 green has the same value as 10_f yellows, then another possibility is that she has 1 green, 11_f yellows, and 4 blues.

(There are lots of other possibilities as well.) □

2.4 Number Lines

In the previous section we worked with number lines for the first time. Students' comfort level working on number lines will impact their number sense as well as their conceptual understanding of many future mathematical topics. Therefore we will take an aside to explore number lines and discuss some subtleties that the reader may not have thought about when working with number lines. These subtleties, if missed by students, could greatly impact their facility with number lines.

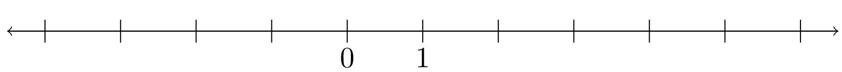
Consider the following question. Where does the number 2 appear in the number line below?



The reader may recognize this as a trick question, or better yet a really easy question because no matter where you place 2 it could be correct since there is no other information given.

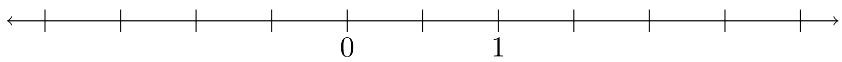
Question 2.4.1. What information would need to be given about the above number line so that there is only one correct tick mark on which to place the number 2?

We now have some more information on the number line below, so we will address the question again. Where does the number 2 appear in the number line below?

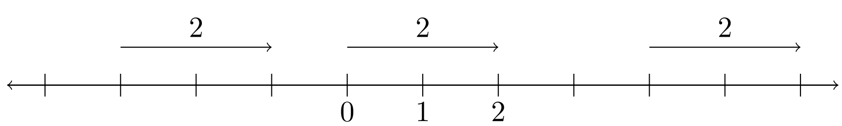


The reader may be thinking that there is only one correct answer to the question now, but here is where the subtlety of what we asked comes into play. Notice that in Question 2.4.1 we used very specific language about placing 2 on a tick mark. Asking what tick mark should be labeled with a

2 is different than asking where 2 appears in the number line. How is that? Well, let's think about how we would place 2 on a tick mark. On the number line above, we should all agree that the tick mark directly to the right of 1 should be labeled with a 2. On the other hand, what if we had the number line below instead. What tick mark would be labeled 2 now?



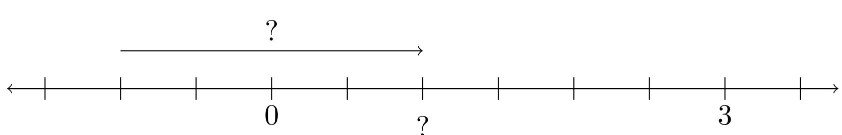
We now see that the tick mark directly to the right of 1 would not be labeled 2. Why is that? Well...the issue here is *distance*. We place 2 on a specific tick mark because it is a distance of 2 from 0. In other words, we label each tick mark with its distance from 0. However, there are lots of distances of 2 on the number line. For example the distance from 3 to 5 is 2. Therefore when we ask where the number 2 appears in a number line, the answer is that it can appear in infinitely many different places. The subtlety we need to keep in mind is when we mark a spot on the number line it is defined in terms of the distance from 0. Therefore each tick mark can only be labeled by one value. However, the value shows up as a distance all over the number line. An example of this is shown below.



Therefore on any number line the number 2 will be a label on exactly one tick mark. However, it can also show up in infinitely many places as an arrow (pointing to the right)* with a length of 2.

*Note: When students are only working with whole numbers, the arrows are often written just as line segments or even rainbows because direction is not of importance until we get to negative numbers. Within this book, we will often use rainbows to indicate our work on the number line when working with whole numbers but will shift to arrows once we start working with negative numbers. This choice is based partly on the motor skills of young students. If we want them to use the number line at an early age we don't want their motor skills to interfere. For some, drawing rainbows may be easier than attempting to draw straight lines, let alone arrows. In addition, work on the number line may involve multiple steps that would be hard to distinguish if straight lines were used for each.

Example 2.4.2. On the number line below, what should the label on the tick mark be, and what should the label on the arrow be?



Answer. The tick mark is labeled 1 and the arrow is labeled 2.

2.5 Addition

Now that we understand the Fenland number system we can begin to learn how to operate with these numbers. Following the elementary school curriculum we will begin with addition. We first must think of what addition actually means. When we want to teach a Kindergartener a problem like $2+1$ we might say something like this: If I have 2 apples and you have 1 apple, how many apples do we have in all? We will use this idea of "all together" for addition. With this understanding of addition, we can see right away that the order of the numbers does not matter because we are just lumping everything into one big pile. Thus $2+1$ and $1+2$ are going to have the same result. We call this the commutative property of addition, and we will address properties more in depth later.

Let's begin our exploration of addition by doing an example with small numbers.

Example 2.5.1. Compute $3+4$.

Possible Solution. Let's try a picture to represent this. We have 3 stars in one pile and 4 stars in another pile, so how many stars do we have in all? Counting up all the stars we see that we have 12_f stars (fen two stars).

⋆	⋆	⋆		⋆	⋆	⋆	⋆
1	2	3		4	10	11	12

Thus $3+4=12_f$. □

Suppose we didn't want to draw pictures. How might a student answer this problem without writing anything down? One possible way is to imagine the pile of 4, and keep that in their head, then counting on from 4 the 3 from the second pile. An example of using this method is shown in the video below for the problem $4+3$ and the problem 23_f+14_f. Notice that to use this method students must be comfortable with recognizing on their hands when they have added the correct amount.

Beginning to Add Video

Technically, we can now add any two numbers. However, if we use one of the above methods to compute 234_f+42_f, then we will be counting for quite a while. Therefore, for a problem like this we need to find a quicker way to find this sum. Let's first consider this problem using Fenland money.

Question 2.5.2. Look back at Examples 2.2.5, 2.2.7, 2.2.9, and 2.2.11. Do any of those stories model an addition problem? What addition problem is being modeled by that story? What is the answer to the addition problem?

Example 2.5.3. What addition problem is modeled by the story problem below?

> Joe has 2 greens, 4 yellows, and 1 blue. Jane has 4 greens and 4 blues. How much money do they have all together?

Answer. Joe has 241_f dollars and Jane has 404_f dollars, so the addition problem we are modeling is $241_f + 44_f$.

(Note: We could now work with the money to find the answer to the addition problem, but in this case we were just looking for the addition problem itself.) □

Fenland money gave us a method to intuitively figure out the sum of two rather large numbers. Let's explore a few more intuitive methods to computing sums.

Example 2.5.4. Compute $234_f + 42_f$ using Base Fen Tiles.

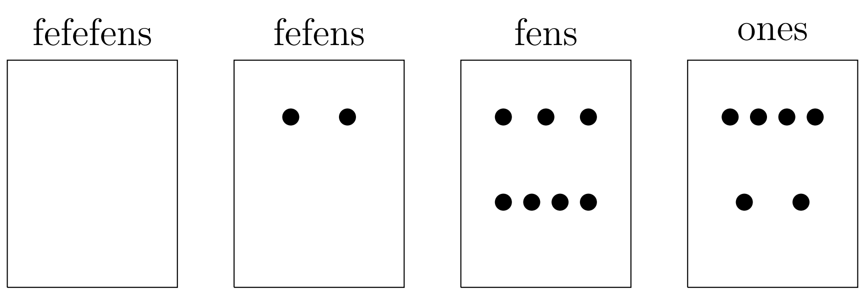
Possible Solution. See the video below. □

Adding Using Base Fen Tiles Video

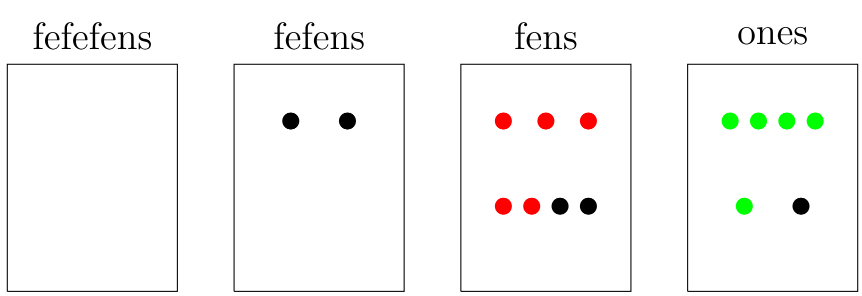
Now let's consider what we did with this strategy. We still used the "all together" idea, but we kept our place values in mind. In particular, we determined how many ones we had all together, and then made any exchanges if possible. Next, we determined how many fens we had all together, and then made any exchanges if possible, and so on. To really solidify this idea of using place value with all together, let's look at how we could do this problem with place value cards.

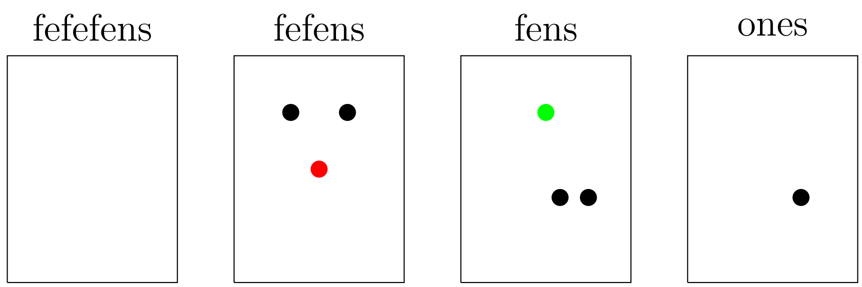
Example 2.5.5. Compute $234_f + 42_f$ using Place Value Cards.

Possible Solution. We want to represent this addition problem in the place value cards. Since addition means all together, we want to represent both numbers in the cards and figure out how much we have all together.



Now we make whatever exchanges we can. In particular we trade fen ones for one fen. (This is marked in green in the diagrams below.) We trade fen fens for one fefen. (This is marked in red in the diagrams below.)





Thus we see we are left with 3 fefens, 3 fens, and 1 one, so $234_f + 42_f = 331_f$. □

We would now like to take the examples we did above and think about how to use the same ideas but without using any manipulatives. In other words, it may be helpful to come up with a method to add two numbers that mimics what we did above, but doesn't require any hands on manipulatives or pictures. In both methods we were just figuring out how much we had in each place value, being sure to make exchanges if possible. This leads us into what is called the "Standard Algorithm for Addition". An example of this is shown below.

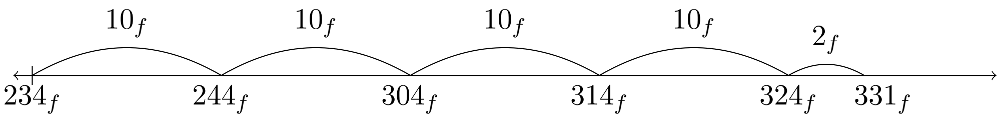
$$\begin{array}{cccc} & 1 & 1 & \\ & 2 & 3 & 4_f \\ + & & 4 & 2_f \\ \hline & 3 & 3 & 1_f \end{array}$$

We call this method the standard algorithm for addition because it is the most common method used in the United States. This is a method you will be teaching your students. Just because we call it the standard algorithm doesn't mean it is the only way, or even necessarily the best way, to add two numbers. We will discuss standard algorithms for the other operations as well, so keep in mind "standard algorithm" just means the most common process (in the U.S.).

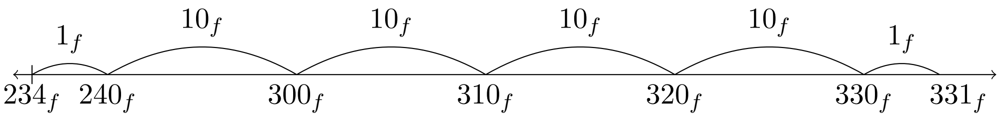
The important thing to notice about the standard algorithm for addition is that we understand why it works using the idea of all together and our understanding of place value. If we truly understand the standard algorithm then we should be able to see the connections between how we added with money, place value cards, base fen tiles, and the standard algorithm.

Question 2.5.6. In the example above, look at the 1 above the 2. What would that 1 correspond to in a money problem? in the place value cards? in the Base Fen Tiles?

As mentioned above, the standard algorithm is not necessarily the best way to compute a sum. In a previous section we discussed the importance of student's recognizing that numbers (and consequently the operations upon those numbers) are flexible. For the addition problem, $234_f + 42_f$, a student could just as easily use a number line to quickly compute this sum. In particular, they could imagine themselves sitting at the number 234_f on the number line and they will need to count on from there 42_f. They might add 42_f by adding fen, then fen, then fen, then fen, then 2. This method is shown below.



A student who uses the method above must be comfortable with jumps in place value. For example, easily recognizing that adding a fen to 244_f will result in 304_f. Some students are not comfortable with that yet, so an intermediate step may be to break 42_f a little differently so that those tricky spots can be avoided. An example of this is shown below.



This method allowed the student to get to a more manageable number of 240_f before adding fens. The two methods we have seen on the number lines above are common strategies students will use. The methods involved either adding a fen or adding something to get to fen. In the U.S. system this corresponds to adding a ten or adding a number that gets you to a ten. When adding a ten we will call that a **jump of ten**. When adding a number to get to a ten we will call that a **jump via ten**. Correspondingly for Fenland we will say a jump of fen or a jump via fen.

Notice that when using the number lines above, they started out unmarked and the students decided what numbers to place on the number line and how to use them. This is often referred to as an **open number line** because students are not restricted by tick marks or number placement. Often times these number lines are not drawn to scale. Open number lines allow students to be flexible with their numbers and use them as they see fit.

2.6 Subtraction

Consider the following problems.

$$14_f - 3_f \qquad 14_f - 12_f \qquad 32_f - 24_f \qquad 32_f - 4_f$$

Think about how you would compute each of them. There are lots of possibilities here, but let's discuss at least a few of them.

Example 2.6.1. Compute $14_f - 3_f$.

Possible Solution. We could put up 14_f fingers, then put 3 down. This would leave us with 11. Therefore $14_f - 3_f = 11_f$. □

Example 2.6.2. Compute $14_f - 12_f$.

Possible Solution. We could put 12_f in our head and count on our fingers until we get to 14_f. So 12_f in our head, one finger up (13_f), another finger up (14_f). Since we put two fingers up, $14_f - 12_f = 2_f$. □

Example 2.6.3. Compute $32_f - 24_f$.

Possible Solution. Again, we could put 24_f in our head, and count on our fingers until we get to 32_f. We would get that $32_f - 24_f = 3_f$. □

Example 2.6.4. Compute $32_f - 4_f$.

Possible Solution. We could put 32_f in our head, and count backward 4 places. So 32_f in our head, then 31_f (one), then 30_f (two), then 24_f (three), then 23_f (four). So $32_f - 4_f = 23_f$. □

Notice in the above problems, we were thinking about subtraction in two different ways. In examples 2.6.1 and 2.6.4, we were removing the smaller amount from the bigger amount. We call this the **take away method**, because we are computing the subtraction problem by taking away a certain amount. However, in examples 2.6.2 and 2.6.3, we were figuring what we needed to add on to the smaller number in order to get the bigger number. We call this the **missing addend method** because we are trying to figure out the missing number we need to add. In our examples right now we are always subtracting a smaller number from a bigger number. However, we are eventually going to be working with negative numbers so we will be able to subtract bigger numbers from smaller numbers. So that our methods work in that setting as well, let's restate our two methods without using the words bigger or smaller.

- To use the **take away method** to compute $a - b$, we will remove the amount b from the amount a. The amount left is the answer to $a - b$.
- To use the **missing addend method** to compute $a - b$, we will figure out what we need to add to b in order to get a. The amount we need to add is the answer to $a - b$.

At this point we need to take an aside to discuss definitions in mathematics. Mathematics is built on a small number of definitions and everything is built logically from those definitions. For example, in the previous section we defined what the symbol $+$ means. We defined $a + b$ to be the value of a and b all together. From that definition, it was clear that the order did not matter. Therefore we did not define addition to be commutative but rather we logically deduced that addition was commutative as a result of the definition of addition. Because definitions play such an important role in mathematics we technically are at a bit of a sticking spot because it seems with the two methods discussed above we have defined the subtraction symbol in two different ways. So let's be clear about definitions. The mathematical definition of the symbol $-$ is that it is the inverse operation of addition, in other words the missing addend method. The definition of subtraction is stated more precisely below.

Definition 2.6.5. $a - b$ is the number that when added to b results in a.

With the definition of subtraction in hand, we now need to logically deduce that the take away method will give us the same answers as the definition. For example, suppose $a - b = c$. According to the definition that means $b + c = a$. In other words, a is made up of b and c. However, if a is made up of b and c and we now remove b, then we are left with c. Thus using the take away method we see that $a - b$ must equal c as well. We now know these two methods are equivalent and will use whichever one we see fit without any further comment.

Question 2.6.6. Look back at Examples 2.2.5, 2.2.7, 2.2.9, and 2.2.11. Do any of those stories model a subtraction problem? What subtraction problem is being modeled by that story? What is the answer to the subtraction problem?

As with addition, we will develop our understanding and facility with subtraction by looking at multiple ways to compute an answer. (The answer to a subtraction problem is called a **difference**.) We will need to eventually get to the point where we understand the Standard Algorithm for Subtraction. However, as a reminder, just because we call it the standard algorithm doesn't mean it is the best. Therefore let's explore subtraction a bit.

Question 2.6.7. Which of the following problems do you feel you could do without paper and pencil?

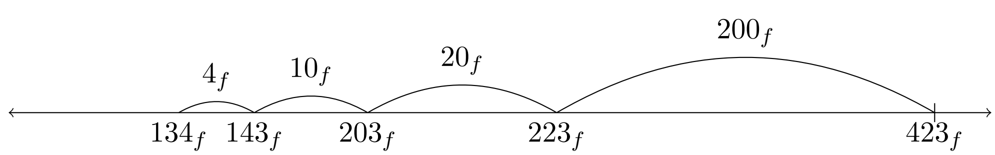
$$21_f - 4_f \qquad 342_f - 3_f \qquad 423_f - 414_f \qquad 423_f - 243_f \qquad 423_f - 234_f$$

Once you have answered the question above and found your level of comfort computing these answers in your head, I encourage you to stretch your brain even more and think about how any of these problems can be done without paper and pencil. Let's spend some time working on $423_f - 234_f$ using multiple methods and finally ending with the standard algorithm.

We will start with the number line. Look back at the examples that were done in your head at the very beginning of this section. When we computed the answer what we did looked very different depending on whether we were thinking of subtraction as missing addend or as take away. The same will be true if we are working on the number line, so let's look at an example of each.

Example 2.6.8. Use the take away method and a number line to compute $423_f - 234_f$.

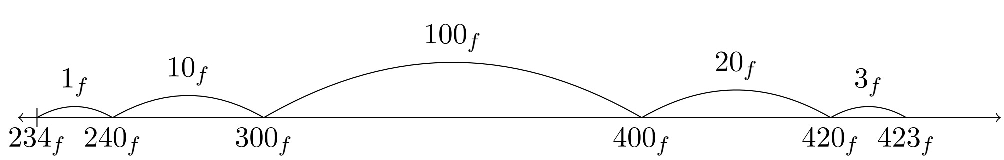
Possible Solution. If we are going to use the take away method, then we want to start at 423_f and take away 234_f. On the number line that means we want to mark 423_f on the number line and go backwards a distance of 234_f. One possible way to do this is shown below.



After going back a distance of 234_f we landed on 134_f. Therefore $423_f - 234_f = 134_f$. □

Example 2.6.9. Use the missing addend method and a number line to compute $423_f - 234_f$.

Possible Solution. If we are going to use the missing addend method, then we want to start at 234_f and figure out what we need to add in order to reach 432_f. One possible way to do this is shown below.



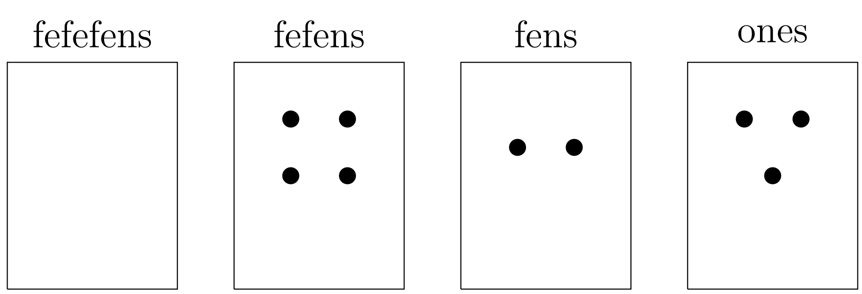
In order to get from 234_f to 423_f we made five jumps. Totaling those five jumps we get 134_f. In other words, starting at 234_f we added a total distance of 134_f to get to 423_f. Therefore $423_f - 234_f = 134_f$. □

It is important that students have methods that allow them to choose how they want to view the numbers. Deciding how to break down a number in a useful way develops their number sense. A student's number sense is an important contributor to their overall understanding of mathematics. In the previous two problems, not only were we flexible with the numbers themselves, we also had the option of how to approach the subtraction problem. Students often favor one method over the other, so it is important as teachers to encourage their thoughtfulness on deciding what method to use. As we saw with previous examples, one method may be better than the other in certain settings.

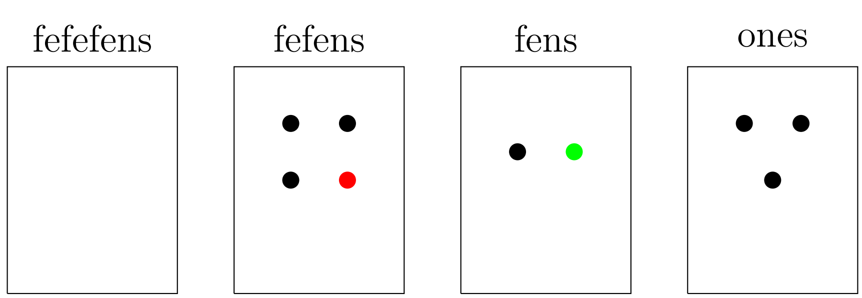
We will now shift to the more place value structured approaches as we head toward an understanding of the Standard Algorithm for Subtraction.

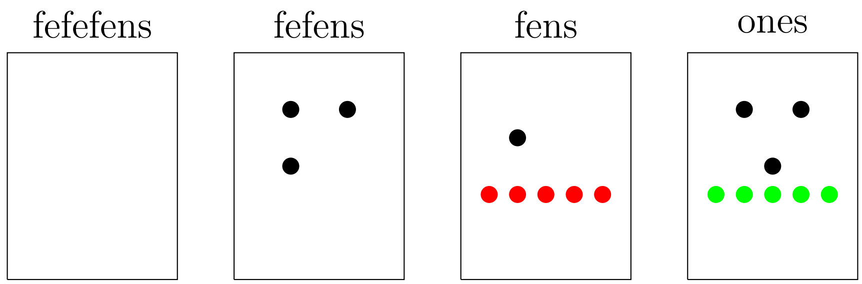
Example 2.6.10. Use place value cards to compute $423_f - 234_f$.

Possible Solution. To compute this difference, we will use the take away method. Therefore we must first represent the number 423_f on the place value cards.

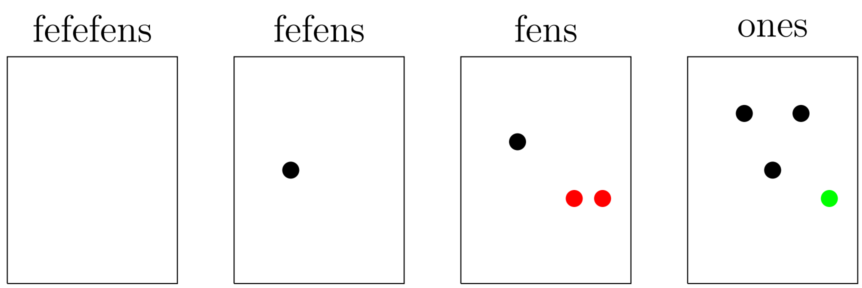


We now need to take away 234_f, or in other words take away 2 fefens, 3 fens, and 4 ones. There are exchanges we need to make before we can do that, so we exchange one fen for fen ones (shown in green) and one fefen for fen fens (shown in red).





Now we have enough in each place value to take away 2 fefens, 3 fens, and 4 ones.



Therefore we are left with 1 fefen, 3 fens, and 4 ones, so $423_f - 234_f = 134_f$. □

Question 2.6.11. What would the work on the place value cards look like if we instead used the missing addend method?

The reader should explore the take away method for $423_f - 234_f$ using base fen tiles and money as well. Connections between the three models should be evident and is what leads us to the standard algorithm.

Let's think about what we did above and try to figure out a way to compute this difference without having to draw a picture. First let's notice that, as with adding, we concentrated on one place value at a time, so we want to be sure that we are working with the same place values. Thus we will begin by stacking the numbers so that we insure that we are working with like place values.

$$\begin{array}{rrrr} & 4 & 2 & 3_f \\ - & 2 & 3 & 4_f \\ \hline \end{array}$$

Thinking back to the place value cards we first exchanged one fen for fen ones. This left us with 1 fen and a total of 13_f ones. We show this below.

$$\begin{array}{rrrr} & & 1 & 13_f \\ & 4 & \cancel{2} & \cancel{3} \\ - & 2 & 3 & 4_f \\ \hline \end{array}$$

Now we can subtract 4 from 13_f, which is 4.

$$\begin{array}{rccc} & & 1 & 13_f \\ & 4 & \not{2} & \not{3} \\ - & 2 & 3 & 4_f \\ \hline & & & 4 \end{array}$$

Again, thinking back to the place value cards, we exchanged one fefen for fen fens. This left us with 3 fefens and a total of 11_f fens. We show this below.

$$\begin{array}{rccc} & 3 & 11_f & 13_f \\ & \not{4} & \not{2} & \not{3} \\ - & 2 & 3 & 4_f \\ \hline & & & 4 \end{array}$$

Finally, we can take 3 away from 11, and we can take 2 away from 3. Thus we get our final answer.

$$\begin{array}{rccc} & 3 & 11_f & 13_f \\ & \not{4} & \not{2} & \not{3} \\ - & 2 & 3 & 4_f \\ \hline & 1 & 3 & 4_f \end{array}$$

Using the method employed in the place value cards, we have developed the Standard Algorithm for Subtraction. For each of the numbers that show up in the standard algorithm it should be clear where that number appears in the place value cards (or money or base fen tiles) and vice versa.

Question 2.6.12. The standard algorithm mimics what we did with the place value cards, but it is not quite exactly the same. What was different about the work in the place value cards compared to the work in the standard algorithm as shown?

2.7 Multiplication (Part I)

Students are now able to add and subtract, so they can figure out the answer to a problem like $3+3$. Originally students see this as an addition problem, but eventually they are introduced to multiplication, so they may now view this problem as 2 groups of 3. In other words, $2 \times 3 = 3+3$. Similarly 3×4 means 3 groups of 4, and so can be computed by finding the sum $4+4+4$. Since $4_f + 4_f + 4_f = 22_f$, then the answer to the multiplication problem $3_f \times 4_f$ is 22_f. The answer to a multiplication problem is called a **product**.

As we did with subtraction, we want to be specific about what our definition of multiplication is. Even though we are only working with whole numbers right now, we will eventually be working with fractions for example, so we want to make sure the language we use applies in that setting if possible.

Definition 2.7.1. $a \times b$ is defined to be the amount in a groups if each group holds a value of b.

More succinctly, $a \times b$ means the amount in a groups of b. Notice that if a is a natural number then this definition is equivalent to the repeated addition shown below. While working in the whole numbers we will use the definition and the repeated addition form interchangeably without further comment.

$$a \times b = \underbrace{b + b + \cdots + b}_{a \text{ times}}$$

Example 2.7.2. Compute $3_f \times 4_f$.

Possible Solution. We need to determine how much there will be in 3 groups of 4.

$$3 \times 4 = 4 + 4 + 4 = 22_f$$

□

Example 2.7.3. Compute $4_f \times 100_f$.

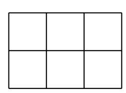
Possible Solution.

$$4 \times 100_f = 100_f + 100_f + 100_f + 100_f = 400_f$$

□

Question 2.7.4. Look back at Examples 2.2.5, 2.2.7, 2.2.9, and 2.2.11. Do any of those stories model a multiplication problem? What multiplication problem is being modeled by that story? What is the answer to the multiplication problem?

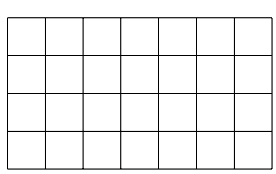
Sometimes it is helpful for students to have a visual to help them figure out the answer to the multiplication problems. We often use arrays to represent a multiplication problem. When we want to compute the multiplication problem 2×3, we can think of this as two groups of three, or pictorially as 2 rows of 3. We can represent this by the array shown below.



Students now have a visual representation of the multiplication problem. To get the answer to the multiplication problem they need only count up the number of (small) squares in the array.

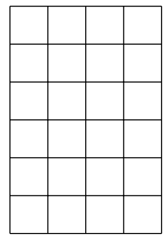
Example 2.7.5. Draw an array to represent $4_f \times 12_f$.

Possible Solution. Since $4_f \times 12_f$ means 4 groups of 12_f, we want an array that has 4 rows with 12_f in each row.

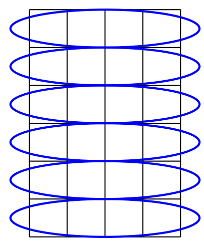


□

Example 2.7.6. What multiplication problem is represented by the array below?



Possible Solution. We have 11_f rows with 4 in each row. Therefore this array represents 11_f groups of 4, as shown below.



Therefore this array is a representation of the multiplication problem $11_f \times 4_f$. □

Question 2.7.7. What other multiplication problem could the above array represent?

2.8 Properties

Now that we've learned a few operations (namely addition, subtraction, and multiplication), we need to take a side step and discuss properties of these operations. There are three main questions we want to answer. Does order matter? Does grouping matter? How do two different operations interact? These three questions corresponds respectively to the commutative property, associative property, and distributive property. Let's state each of these properties formally without specifying a known operation.

Definition 2.8.1. The **commutative property of** $\star$ states that for any numbers a and b

$$a \star b = b \star a$$

Definition 2.8.2. The **associative property of** $\star$ states that for any numbers a, b, and c

$$a \star (b \star c) = (a \star b) \star c$$

Definition 2.8.3. The **left distributive property of** $\star$ **over** $\circ$ states that for any numbers a, b, c

$$a \star (b \circ c) = (a \star b) \circ (a \star c)$$

Definition 2.8.4. The **right distributive property of** $\star$ **over** $\circ$ states that for any numbers a, b, c

$$(b \circ c) \star a = (b \star a) \circ (c \star a)$$

Let's consider each of these properties in relation to the operations we have worked with so far.

Example 2.8.5. State the commutative property of multiplication.

Answer. The commutative property of multiplication states that for any numbers a and b

$$a \times b = b \times a$$

□

Just because we can state a property does not mean the property is true. Therefore we want to have language that distinctly separates those two ideas. We can write down an equation for any of the properties for any operation. When we do this we say we are stating the property. We will most likely then want to decide if that equation is true no matter what numbers we use. If the equation is in fact true for all values, then we say that the property holds. Let's consider the commutative property of multiplication. We stated the commutative property of multiplication above. Does the commutative property of multiplication hold? The reader probably remembers from their elementary days that the order does not matter in a multiplication problem, so we would say the commutative property of multiplication holds. But can you give a logical explanation as to why?

Question 2.8.6. How could we convince someone who knows the definition of multiplication that the commutative property of multiplication holds?

To conclude that a property holds we need to show the equation is true for all values we choose. Now, since we obviously cannot actually check the equation for every single number that means we would need to find a logical argument that shows the equation is true for all numbers. That is a common theme in mathematics. How do we show something will always be true if we can't actually check every possibility? We will work on this continuously throughout this course and the next.

On the other hand, let's think about how we would show a property does not hold. That would mean it is not the case that the equation is true for all numbers. Therefore if we can find just one set of numbers that makes the equation untrue, then we have shown the equation does not work for all numbers. When we find an example that makes the statement untrue it is called a **counterexample**. So in some sense it is much easier to prove a property does not hold because we just need to find a single counterexample.

Example 2.8.7. State the commutative property of subtraction.

Answer. The commutative property of subtraction states that for any numbers a and b

$$a - b = b - a$$

□

Example 2.8.8. Does the commutative property of subtraction hold?

Possible Solution. Consider $a = 4$ and $b = 1$. $a - b = 4 - 1 = 3$. However $b - a = 1 - 4$, and $1 - 4$ is a subtraction problem that does not have an answer in the whole numbers (which are the only numbers we are working with so far). Therefore the answer to $4 - 1$ is not equal to the answer to $1 - 4$. In other words, $4 - 1 \neq 1 - 4$. Therefore we have found a counterexample, and so the commutative property of subtraction does not hold. □

As stated above, if we are trying to prove that a property holds we need to show it will be true for *all* numbers, so one example is not sufficient to prove a property true. An example, namely a counterexample, is sufficient to prove a property does not hold. We should be careful about counterexamples and properties not holding for fear of overstating conclusions. If a property does not hold, then that just means we have found instances where the equation is not true. It does *not* mean the equation is never true. It is helpful to think of three categories for any given equation, namely always true, sometimes true, or never true. For example, the equation in the statement of the commutative property of multiplication is always true.

Question 2.8.9. Into which category does the equation in the statement of the commutative property of subtraction fall?

So far we have only explored the properties that involve a single operation. The distributive property explores the interaction between two operations. There are several distributive properties, but usually when someone says "the distributive property" they mean the left distributive property of multiplication over addition.

Example 2.8.10. State the left distributive property of multiplication over addition.

Answer. The left distributive property of multiplication over addition states that for all numbers a, b, and c

$$a \times (b + c) = (a \times b) + (a \times c)$$

□

Let's "play" with this property a bit. When mathematicians say they are going to "play" with it, they often mean they are going to look at some examples to get a better feel for what the property is saying. After looking at some examples, we may have a better idea as to whether or not the property holds. Now, you may remember from your elementary days that the distributive property of multiplication over addition does in fact hold, but let's go through this process of play anyway. When you are faced with an unfamiliar property or statement, this process of "play" can prove to be very useful.

To play with this property, we need to pick some numbers. How about $a = 3_f$, $b = 4_f$ and $c = 2_f$. We need to decide if, for our chosen values, the equation in the statement of the left distributive property of multiplication over addition is true. The equation is shown in the answer to Example 2.8.10.

Plugging our values into the left hand side of the equation, we get the following.

$$\begin{aligned} a \times (b + c) &= 3_f \times (4_f + 2_f) \\ &= 3_f \times 11_f \\ &= 11_f + 11_f + 11_f \\ &= 33_f \end{aligned}$$

So for our values, the left hand side of the equation is equal to 33_f. Now let's explore the right hand side.

$$\begin{aligned}(a \times b) + (a \times c) &= (3 \times 4) + (3 \times 2) \\ &= 22_f + 11_f \\ &= 33_f\end{aligned}$$

For our values, the right hand side of the equation also equals 33_f. So for our chosen values the equation is in fact true. Note that this does not necessarily mean the property holds because we have only checked one example. However, as we check more examples (and the reader is encourage to check at least one more on their own) we will see that the left hand side and right hand side are equal for every example we try. Now we suspect the property does hold, but how do we know there's not some example out there where it won't work. That is why we need to provide an argument that shows no matter what numbers we pick, the equation wiil be true.

Question 2.8.11. How could we use arrays to justify that the left distributive property of multiplication over addition holds?

We have been focusing on the *left* distributive property of multiplication over addition, namely $a \times (b+c) = (a \times b) + (a \times c)$. Do we then need to spend the same amount of time discussing the *right* distributive property of multiplication over addition? In short, no! The right distributive property of multiplication over addition states $(b+c) \times a = (b \times a) + (c \times a)$, so the only difference is that the a is on the right hand side. However, note that we are *multiplying* by a and since multiplication is commutative we can easily deduce that the right distributive property of multiplication over addition holds from the fact that the left property holds. We give that argument below.

Out of pure laziness, for the remainder of this paragraph we will use the term "distributive property" to mean the distributive property of multiplication over addition. Once we have an argument that justifies the left distributive property (which you were asked to do in Question 2.8.11), we know that $a \times (b+c) = (a \times b) + (a \times c)$ for all values of a, b, and c. However, by the commutative property of multiplication we know that $a \times (b + c) = (b + c) \times a$. Similarly $a \times b = b \times a$ and $a \times c = c \times a$. Making these substitutions we see that if the left property is true, then the right must be as well. These substitutions are shown below.

$a \times (b + c) = (a \times b) + (a \times c)$

$\downarrow \qquad \downarrow \qquad \downarrow$

Replace each expression with an equivalent one using the commutative property of multiplication

$\downarrow \qquad \downarrow \qquad \downarrow$

$(b + c) \times a = (b \times a) + (c \times a)$

Therefore we have shown that if the left distributive property of multiplication over addition is true, then the right one must be as well. This will not always be the case however, because not all operations are commutative and that was what allowed us to make the substitutions.

Question 2.8.12. Does the distributive property of multiplication over addition hold if we have more than two numbers added in the parenthesis? For example, is it always true that $a \times (b+c+d) = (a \times b) + (a \times c) + (a \times d)$?

Example 2.8.13. State the left distributive property of division over addition and the right distributive property of division over addition.

Answer. The left distributive property of division over addition states that for all numbers a, b, and c

$$a \div (b + c) = (a \div b) + (a \div c)$$

The right distributive property of divison over addition states that for all numbers a, b, and c

$$(b + c) \div a = (b \div a) + (c \div a)$$

□

Notice that since division is not commutative, then we will have to explore each of the above properties separately. In other words, just because one is true does not necessarily mean the other is true as was the case with multiplication.

Question 2.8.14. Do either of the properties stated in Example 2.8.13 hold?

2.9 Multiplication (Part II)

Equipped with the definition of multiplication and our knowledge of properties, we can now begin to compute more difficult multiplication problems. Our goal is to eventually have a fairly quick way to compute a problem like $234_f \times 43_f$, for example. Keep in mind that we do have a way right now to compute this, namely we could add 234_f up four fen three times. Though that is doable, it doesn't sound very efficient. Our goal is to come up with an efficient method for computing $234_f \times 43_f$.

It is worth taking a minute and thinking about what could make us more "efficient" when solving multiplication problems. By the time students get to multiplying large numbers, they have been working with single digit multiplication for quite some time. In fact, fluency with single digit multiplication is expected (though definitely not necessarily attained) by this time. Therefore, if we can figure out a way to multiply large numbers in a way that only requires us to know single digit multiplication, then that may lead us toward a more efficient method.

I would be remiss if I didn't add just a small note about how students attain fluency in single digit multiplication. There is a whole pedagogical discussion that could be had about helpful and harmful methods that are used in pursuit of fluency, and I encourage you to educate yourself on this topic if you find yourself teaching this topic. However, in order to purse the mathematics, we will not have that discussion within this text.

We can now articulate what we think we need in order to become more efficient multipliers.

Our mission (should we choose to accept it):
Multiply any two numbers using only single digit multiplication.

Challenge accepted! We will build a method step by step, starting with simpler examples and expanding on them.

Example 2.9.1. Compute $4_f \times 300000_f$.

Possible Solution. Using the definition of multiplication we just need to add 100000_f four times.

$$\begin{array}{rrrrrrr} & 3 & 0 & 0 & 0 & 0 & 0_f \\ & 3 & 0 & 0 & 0 & 0 & 0_f \\ & 3 & 0 & 0 & 0 & 0 & 0_f \\ + & 3 & 0 & 0 & 0 & 0 & 0_f \\ \hline & 22 & 0 & 0 & 0 & 0 & 0_f \end{array}$$

Thus $4_f \times 300000_f = 2200000_f$ □

Question 2.9.2. If we changed the 300000_f in the above example to 30000000_f, how would the answer to the multiplication problem change?

Question 2.9.3. If instead we changed the 4_f in the above example to 12_f, how would the answer to the multiplication problem change?

After answering the above two questions, we should be able to make a general statement about multiplication problems of this form. But before we do so we should be careful about what form we are actually looking at here. The first number, a, could be any natural number. The second number in the multiplication problem however has any number in front but then is followed by a certain number of zeros. We will write $b000\ldots000$ to represent the latter type of number. To be clear then a is any natural number, b is any natural number, and the zeros mean there is some number of zeros following b. For example, in the previous example $a = 4$, $b = 3$ and the number of zeros is 5.

We would like to make a general statement about what $a \times b000\cdots000$ will equal. At this point the reader should be able to justify what to do with the zeros and why. However, the number in the answer that goes in front of the zeros may not be clear quite yet. If needed, do a few more examples so that a generalization about the number in front can be made.

Question 2.9.4. What does $a \times b000\cdots000$ equal?

From this point foreword we will assume the reader has answered the question above (with justification). We will therefore assume this result has been proven and can be used as we proceed. Let's now think about how the shortcut we have figured out in the setting above applies to problems of the form $a000\cdots000 \times b000\cdots000$. Perhaps we should start with a specific multiplication problem.

Question 2.9.5. How can we apply the shortcut we found above to compute $300_f \times 4000000_f$? (Hint: It's okay to use the shortcut more than one time.)

Now that we have answered the question in a specific setting, we should think about how our work generalizes to any multiplication problem of this form.

Question 2.9.6. What does $a000\cdots000 \times b000\cdots000$ equal?

In answering the above question, we have found a way to multiply certain large numbers that is more efficient than solely relying on the definition of multiplication. However, if a or b themselves are not a single digit, then we still have some more work to do. Remember, to be efficient, we would like to only have to do single digit multiplication. Therefore, we need to explore how to conveniently break up a number, keeping our goal of single digits in mind.

Example 2.9.7. Break the number 234_f up so that it is the sum of numbers of the form $b000\cdots000$ where b is always a single digit.

Possible Solution. $234_f = 200_f + 30_f + 4_f$

It is now time to remind ourselves of a property we discussed previously. A property that will get us exactly what we need to accomplish our mission. Go ahead, look back at Example 2.8.10 to see what this all important property is. With this property in hand and our work in the previous example, we are ready to take the first step on our mission.

Example 2.9.8. Compute $4_f \times 234_f$.

Possible Solution. We begin by breaking up the number as above. Therefore $4 \times 234_f$ becomes $4 \times (200_f + 30_f + 3)$. Now we can apply the all important distributive property of multiplication over addition. These steps are shown below.

$$\begin{aligned} & 4_f \times 234_f \\ &= 4_f \times (200_f + 30_f + 4_f) \\ &= (4_f \times 200_f) + (4_f \times 30_f) + (4_f \times 4_f) \end{aligned}$$

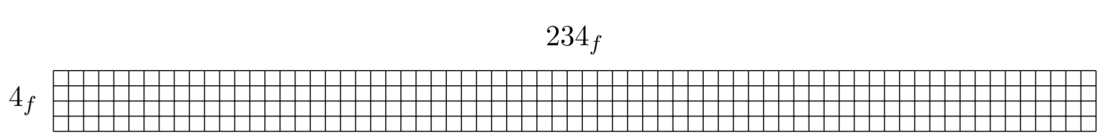
Using our answer to Question 2.9.4, we know that $4 \times 200_f$ is just the answer to $4_f \times 2_f$ followed by two zeros. Since $4_f \times 2_f = 13_f$, then $4 \times 200_f = 1300_f$. We compute similarly for the other two expressions, giving us the following.

$$\begin{aligned} &= (4_f \times 200_f) + (4_f \times 30_f) + (4_f \times 4_f) \\ &= 1300_f + 220_f + 31_f \\ &= 2101_f \end{aligned}$$

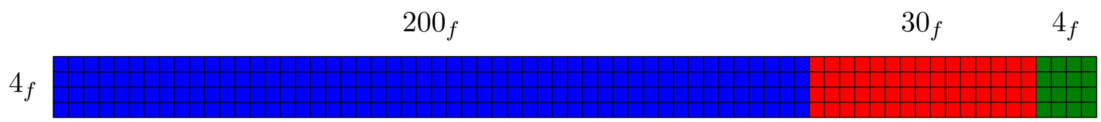
Therefore $4_f \times 234_f = 2101_f$.

Looking back at our work, let's pay attention to what multiplication problems we actually had to solve. We computed $4_f \times 2_f$, $4_f \times 3_f$, and $4_f \times 4_f$. Only single digit multiplication! We are well on our way to completing our mission. Our success relied on the fact that we can think of 234_f as $200_f + 40_f + 3_f$. When we break a number up into its place values like this, we call it **expanded notation**.

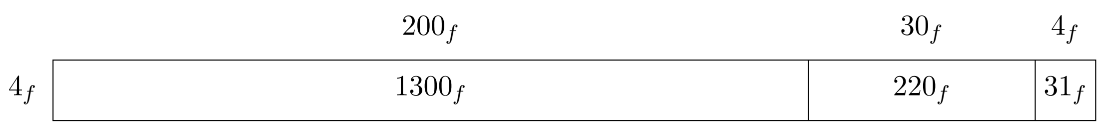
Notice that we used the distributive property of multiplication over addition in these problems. Recall that in the previous section we used arrays to understand the distributive property. Let's use arrays to get a visual of what we did above. Normally if we were going to use an array to represent $4_f \times 234_f$ we would need to have 4_f rows with 243_f squares in each row, which is shown below.



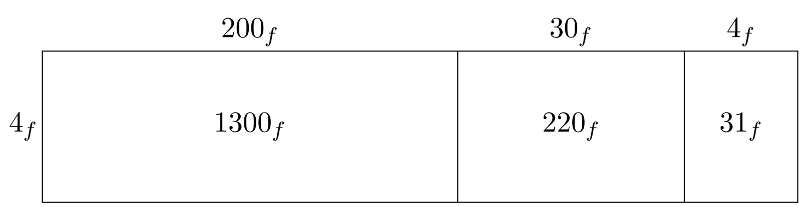
That's a lot of boxes to draw, so we must find a better way to use arrays without having to draw in all of the boxes. Let's first look at the top number in expanded form and break up the array accordingly.



We know $4_f \times 200_f = 1300_f$, so there are 1300_f squares in the blue section. Similarly $4_f \times 30_f = 220_f$, so there 220_f squares in the red section. Finally $4_f \times 4_f = 31_f$, so there are 31_f squares in the green section. And so there are $1300_f + 220_f + 31_f = 2041_f$ squares in all. Notice that we did not actually need to see the squares to determine how many there are, so we can represent this same picture but just remove the squares and colors.



When we draw an array labeled with expanded notation and simply write the number of squares in each area rather than draw all of the squares in, we call this an **expanded array**. Notice that as we removed the squares we kept the scale correct, but in practice when we draw an expanded array we do not necessarily care too much about how accurate the scale is. For example, a perfectly acceptable expanded array for $4_f \times 234_f$ is shown below.



The expanded array gives us a way to represent any multiplication problem visually even if the numbers are rather large. In the example we did above, only one of the numbers needed to be written in expanded form, however we can use this same idea if both numbers are large enough to be expanded. This can be seen in the example below.

Example 2.9.9. Use an expanded array to represent $32_f \times 342_f$, and use it to help find the product.

Possible Solution. We want to think of each of these numbers in expanded notation. So we will think of 32 as $30 + 2$ and we will think of 342 as $300 + 40 + 2$.

	300_f	40_f	2_f
30_f	14000_f	2200_f	110_f
2_f	1100_f	130_f	4_f

Recall that the number in each of the boxes represents how many squares there would be in each of those regions if we actually drew this array to scale with all of the squares. Therefore we can get the total number of squares, and thus the answer to the multiplication problem by adding those numbers up.

$$\begin{array}{cccccc} & 1 & 4 & 0 & 0 & 0 \\ & & 2 & 2 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & 1 & 1 & 0 & 0 \\ & & & 1 & 3 & 0 \\ + & & & & & 4 \\ \hline & 2 & 3 & 0 & 4 & 4 \end{array}$$

Thus $32_f \times 342_f = 23044_f$. $\square$

Though the picture is nice and helps us see what is going on, it does take up a lot of room and is time consuming. Let's begin to streamline this process a little bit. We will redo the problem in the same way, but without drawing a picture. We will call this method the **expanded algorithm for multiplication**. Notice that when we expanded 342_f into $300_f + 40_f + 2_f$ and 32_f into $30_f + 2_f$, we ended up multiplying all combinations of those expansions. Therefore we can just record those in a string below the multiplication problem. This method is shown below to compute $32_f \times 342_f$.

$$
\begin{array}{rrrrrl}
 & 3 & 4 & 2_f & & \\
\times & & 3 & 2_f & & \\
\hline
 & & & 4_f & \leftarrow 2_f \times 2_f \\
 & 1 & 3 & 0_f & \leftarrow 2_f \times 40_f \\
1 & 1 & 0 & 0_f & \leftarrow 2_f \times 300_f \\
 & 1 & 1 & 0_f & \leftarrow 30_f \times 2_f \\
2 & 2 & 0 & 0_f & \leftarrow 30_f \times 40_f \\
1\ 4 & 0 & 0 & 0_f & \leftarrow 30_f \times 300_f \\
\hline
2\ 3 & 0 & 4 & 4_f &
\end{array}
$$

Thus $32_f \times 342_f = 23044_f$.

Example 2.9.10. Use the expanded algorithm for multiplication to compute $42_f \times 413_f$.

Possible Solution.

$$
\begin{array}{rrrrrr}
 & & & 4 & 1 & 3_f \\
 & & \times & & 4 & 2_f \\
\hline
 & & & & 1 & 1_f \\
 & & & & 2 & 0_f \\
 & & 1 & 3 & 0 & 0_f \\
 & & & 2 & 2 & 0_f \\
 & & & 4 & 0 & 0_f \\
+ & 3 & 1 & 0 & 0 & 0_f \\
\hline
 & 3 & 4 & 0 & 0 & 1_f
\end{array}
$$

Therefore $42_f \times 413_f = 34001_f$. □

Our mission has been accomplished. We can now multiply any two numbers by using only single digit multiplication. In addition, the expanded algorithm is grounded in conceptual understanding as it follows directly from the arrays associated with multiplication. The reader may recognize however, that this is not the standard algorithm for multiplication learned in elementary school. Again, we emphasize that just because we have a *standard* algorithm does not mean it is superior. In particular, the method above is a perfectly fine and efficient way to multiply two numbers. However, there is a push to streamline it even more, which is how we arrive at the standard algorithm.

Before watching the video below the reader is encouraged to compute 635×78 using the standard algorithm for multiplication. The video then goes through the **standard algorithm for multiplication** computing 635×78 (which is in the U.S. number system) side by side with $32_f \times 342_f$.

Standard Multiplication Algorithm Video

Although the reader may be comfortable with the standard algorithm for multiplication, it is worth pausing to think about how much stuff is happening within the algorithm and how difficult this could potentially be for students, especially if it is the only method that is taught. If, on the other hand, they have had experience with the expanded algorithm they then have a chance to understand why the multiple steps are occurring. With that in mind, we should make sure we can also make direct connections between the standard algorithm for multiplication and the more conceptual expanded algorithm.

Question 2.9.11. The standard algorithm and the expanded algorithm for $342_f \times 32_f$ are shown side by side. Where do each of the numbers in the standard algorithm appear in the expanded algorithm?

$$\begin{array}{ccccc}
 & & 2_f & 1_f & \\
 & & \not{1}_f & & \\
 & & 3 & 4 & 2_f \\
 & \times & & 3 & 2_f \\
\hline
 & 1 & 2 & 3 & 4_f \\
2 & 1 & 3 & 1 & 0_f \\
\hline
2 & 3 & 0 & 4 & 4
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & 3 & 4 & 2_f \\
 & \times & & 3 & 2_f \\
\hline
 & & & & 4_f \\
 & & 1 & 3 & 0_f \\
 & 1 & 1 & 0 & 0_f \\
 & & 1 & 1 & 0_f \\
 & 2 & 2 & 0 & 0_f \\
1 & 4 & 0 & 0 & 0_f \\
\hline
2 & 3 & 0 & 4 & 4_f
\end{array}$$

2.10 Division

As we saw with subtraction, there are multiple ways to interpret and thus compute the answer to the division problem. Therefore we must again begin with the definition of division, and then justify that the other methods do in fact coincide with the definition. Also analogous to subtraction, division is defined to be the inverse operation to multipliecation.

Definition 2.10.1. $a \div b$ is the number that when multiplied by b results in a.

For example, suppose we want to compute $13_f \div 2_f$. According to the definition we need to figure out what 2 can be multiplied by in order to get 13_f. Thus $2_f \times ? = 13_f$. From our math facts we know that $2_f \times 4_f = 13_f$, thus $13_f \div 2_f = 4_f$.

Example 2.10.2. Use the definition of division to compute $121_f \div 14_f$.

Possible Solution. We need to figure out what the question mark is in the equation $14_f \times ? = 121_f$. Let's try some numbers. Our first guess is 3.

$$\begin{array}{r} {}^{2} \\ 1\;4 \\ \times\quad 3 \\ \hline 1\;0\;2 \end{array}$$

The result was smaller than 121_f, so 3 isn't big enough. Let's try 4.

$$\begin{array}{r} {}^{3} \\ 1\;4 \\ \times\quad 4 \\ \hline 1\;2\;1 \end{array}$$

When we multiplied 14_f by 4_f the result was 121_f. Thus $121_f \div 4_f = 14_f$. □

Since division is defined in terms of multiplication, let's go back to the definition of multiplication. For example, 3×4 was defined to be the amount in 3 groups of 4. However, since multiplication is commutative, we could also think of this as 4 groups of 3. So in any multiplication problem one of the factors represents the number of groups and the other factor represents the amount in each group. The fact that every multiplication problem can be interpreted in one of two ways means any division problem can as well. Let's be more explicit by what we mean here.

Consider the division problem, $22_f \div 3_f$. The answer to this division problem is the number that fills in the question mark in the equation $3_f \times ? = 22_f$. Suppose we interpret the multiplication problem so that 3_f is the number of groups. Then the question mark would have to be the amount in each group and 22_f would be the total amount.

$$\begin{array}{ccccc} 3_f & \times & ? & = & 22_f \\ \downarrow & & \downarrow & & \downarrow \\ \text{(number of groups)} & & \text{(amount in each group)} & & \text{(total amount)} \end{array}$$

Therefore the answer to the division problem $22_f \div 3_f$, in other words the question mark, can be thought of as the answer to the following question. If we have a total of 22_f and we break it into 3 equal groups, how much will be in each group? Therefore in this case the division problem is asking us to fairly distribute an amount of 22_f into 3 groups. When a division problem takes on this role we will call this division by sharing. (We use this name because it should be reminiscent of sharing fairly among 3 friends for example.)

If on the other hand we view the 3_f in the multiplication problem as the amount in each group, we get the following.

$$\begin{array}{ccccc} 3_f & \times & ? & = & 22_f \\ \downarrow & & \downarrow & & \downarrow \\ \text{(amount in each group)} & & \text{(number of groups)} & & \text{(total amount)} \end{array}$$

Therefore the answer to the division problem $22_f \div 3_f$, in other words the question mark, can be thought of as the answer to the following question. If we have a total of 22_f and we make groups of size 3, how many groups can we make? In this scenario we know the group size so we will call this division by grouping.

Thus we can see that any division problem can be viewed in one of these two ways. In particular, when faced with a story problem students will need to recognize one of these scenarios in order to understand that it can be solved via division. We described each of the methods above using an example, but we should state them in general.

Definition 2.10.3. To interpret the division problem $a \div b$ using the **division by sharing method** we would need to take a total amount of a and break it evenly into b groups. The answer to the division problem $a \div b$ is the amount in each of those groups.

Definition 2.10.4. To interpret the division problem $a \div b$ using the **division by grouping method** we would need to take a total amount of a and break it evenly into groups of size b. The answer to the division problem $a \div b$ is the number of groups formed.

Example 2.10.5. The story problem below can be solved by computing $121_f \div 4$. Is the problem an example of division by sharing or division by grouping?

> Joe has 121_f pieces of candy. He wants to share the candy fairly among him and his three friends. How many pieces of candy does each person get?

Possible Solution. If we think about how this would physically happen, Joe would probably take the candy and pass it out one by one into four piles and each friend would get a pile. Therefore Joe is taking 121_f and breaking it into 4 groups, and the question being answered is how many there will be in each group. Therefore this is an example of division by sharing. □

Example 2.10.6. The story problem below can be solved by computing $121_f \div 4$. Is the problem an example of division by sharing or division by grouping?

> Joe has 121_f pencils. He makes pencil bundles by putting a rubber band around four pencils. How many bundles can Joe make?

Possible Solution. Again if we think about physically how this would happen, Joe would take the pile of pencils, make a group of 4 and set it down. Make another group of 4 and set it down. Therefore he is taking 121_f and making groups of size 4, and the question being answered is how many groups will be made. Therefore this is an example of division by grouping. □

Example 2.10.7. The story problem below can be solved by computing $121_f \div 4$. Is the problem an example of division by sharing or division by grouping?

> Joe gets four dollars for each lawn he mows. How many lawns must Joe mow in order to make 121_f dollars?

Possible Solution. This problem is a little more subtle. The total amount Joe will get is 121_f. The way he is going to get that total is by earning 4 dollars each time. In other words, we need to find how many groups of size of 4 will we need to reach our total of 121_f. Therefore this is an example of division by grouping. □

Question 2.10.8. Look back at Examples 2.2.5, 2.2.7, 2.2.9, and 2.2.11. Do any of those stories model a division problem? What division problem is being modeled by that story? Of what method of division is it an example? What is the answer to the division problem?

Beyond just recognizing a division problem in a story, the methods of division and the definition of division affect how students actually compute the answer to a division problem. The answer to a division problem is called a **quotient**. We will consider the same division problem, namely $41_f \div 3_f$, and explore how students would compute the quotient depending on how they are thinking about division.

A student who computes $41_f \div 3_f$ by thinking about the definition of division will be looking for a number to multiply by 3 in order to get 41_f. They may start with their math facts to see if there are any known ones that work, and from there they start adjusting their guess and checking. For example, the student may remember that $3_f \times 4_f = 22_f$, so 4_f is too small. Next they may try 11_f, but $3_f \times 11_f = 33_f$ so 11_f is still too small. Finally they try 12_f, and $3_f \times 12_f = 41_f$. Therefore $41_f \div 3_f = 12_f$.

A student who computes $41_f \div 3_f$ by thinking of division by sharing will need to break 41_f into 3 groups. The video below shows an example of what this might look like.

Division by Sharing Video

A student who computes $41_f \div 3_f$ by thinking of division by grouping will need to break 41_f into groups of size 3. The video below shows an example of what this might look like.

Division by Grouping Video

In the above example of computing using the division by grouping method, let's think about the corresponding arithmetic. The student started with 41_f counters and then took 3 away, and then took 3 more away, etc. Therefore this student could have figured out the answer to the division problem without use of the counters by just repeatedly subtracting 3 until we get 0. For this reason, the division by grouping method is sometimes referred to as repeated subtraction. An example of work a student might show if they are using this approach is shown below.

$$41_f - 3_f = 33_f \quad \boxed{1_f}$$
$$33_f - 3_f = 30_f \quad \boxed{2_f}$$
$$30_f - 3_f = 22_f \quad \boxed{3_f}$$
$$22_f - 3_f = 14_f \quad \boxed{4_f}$$
$$14_f - 3_f = 11_f \quad \boxed{10_f}$$
$$11_f - 3_f = 3_f \quad \boxed{11_f}$$
$$3_f - 3_f = 0 \quad \boxed{12_f}$$

Since we subtracted 3_f a total of 12_f times, then $41_f \div 3_f = 12_f$.

To be clear, repeated subtraction is not a new method of division, but rather it is the arithmetic that would be done if a student is viewing a division problem as division by grouping.

Question 2.10.9. What would a student's work look like if they were computing $41_f \div 3$ on the number line, and they were using division by grouping?

Question 2.10.10. What would a student's work look like if they were computing $41_f \div 3$ on the number line, and they were using division by sharing?

Theoretically a student can now divide any two numbers using one of the many ways described above. However, the issue is that this could take a long time. Consider the problem $342143242 \div 12312$, we probably don't want to guess and check products, and we probably don't want to write out 342143242 things. Therefore we need to find a more efficient way to compute the answer to a division problem. To do this we are going to rely on the division by sharing method. In the video below we are going to solve the following money problem while simultaneously recording our process in a manner that will lead us directly into the process of long division.

> Joe has 4143 dollars. He wants to give each of his 4 friends the exact same pile of money. He is at the bank to make any exchanges he needs. What will Joe give each friend?

Long Division and Money Sharing Video

Using the money idea, we have made sense out of the long division process. Notice that we didn't really need the money, we could have just thought about place value. In particular, we could have performed the money exchanges using place value cards instead. Whenever we said "pink" we could have said fefefens. Whenever we said "green" we could have said fefens, etc. Therefore the long division process is really just division by sharing, where we are sharing place value by place value.

2.11 Exercises

Any problem that is stated in the Fenland system should be completed entirely within the Fenland system unless indicated otherwise.

1. Use base fen tiles to compute the following. Please use pictures and words to describe what you did with the base fen tiles in order to arrive at your answer.
   (a) Three fefen two plus four fen four.
   (b) Two fefen three fen one take away four fen three.

2. A number is said to be even if it is divisible by 2. It turns out that in our number system, we know a number is even if the ones digit of the numeral is 0,2,4,6, or 8. However, the ones digit does not determine evenness in the Fen system. Therefore we need to be sure we are only working with the definition of even. When we say a number is divisible by 2, then using the two methods of division, that means either the number can be broken up evenly into 2 groups (division by sharing) or the number can be made into groups of 2 (division by grouping). In answering the questions below, think carefully and be strategic about which version of "divisible by 2" you use.
   (a) Use the definition of even to determine if fen four is an even number?
   (b) Use the definition of even to determine if three fen one is an even number?
   (c) Without actually drawing out two fefen four fen things, explain how you know two fefen four fen is an even number.
   (d) Without actually drawing out three fefen two fen one things, determine if three fefen two fen one is even?

3. For each of the following, identify the error that the student has made in base ten and describe an analogous error in base fen.
   (a) The student counts: "one hundred and seven, one hundred and eight, one hundred and nine, two hundred, two hundred and one, two hundred and two..."
   (b) The student looks at the numeral 31,407 and says that it is "three million one thousand four hundred and seven."
   (c) The student looks at the number 31,407 and says that it is "three thousand one hundred and forty seven."
   (d) The student subtracts 5 from 302 and gets 207.

4. Jayden has three greens and two blues. Ming has four yellows and four blues.
   (a) Jayden and Ming put all their money together and go to the bank. They make exchanges until they have the fewest number of bills possible. What bills do they have now? Please describe all exchanges you made and how you arrived at your answer.
   (b) How much money do they have all together?

5. Jackson has four greens and two blues in his wallet, and he wants to buy a comic book that costs fen three dollars.

   (a) What bills could Jackson give to the cashier, and what bills will he receive in change? (The cashier alway gives back the fewest number of bills possible.) Please show how you arrived at your answer.

   (b) After Jackson buys the comic book and receives his change, how much money does he have? Please show how you arrived at your answer.

6. Deepak has one pink, three greens, two yellows, and four blues saved up in his dresser drawer that he plans to give to his four children as a gift. He knows that if it is not clear immediately that each child is getting the same amount of money, someone will complain. Deepak is going to take the money to the bank and exchange it for bills that will allow him to give each child the exact same bills. Deepak wants to use as much of the money as possible, but needs to stay fair.

   (a) What bills will each child get? Does Deepak have any bills left over?

   (b) How much money does each child get?

7. (a) Write a Fenland money story that models the sum $302_f + 44_f$.

   (b) Use the money to solve your problem. Be sure to make it clear what was done with the money.

   (c) Use place value cards to compute the sum $302_f + 44_f$.

   (d) Use the standard addition algorithm to compute the sum $302_f + 44_f$. Show all work.

   (e) For **every digit** that shows up in your **work** for Exercise 7d, describe where that digit shows up in your money problem above (problem 7b).

   (f) For **every digit** that shows up in your **work** for Exercise 7d, describe where that digit shows up in the place value cards above (problem 7c).

8. For this problem we are assuming students have just been introduced to subtraction, so they do not have answers to subtraction problems memorized yet.

   (a) Explain how a student who is using the take-away method could compute $12_f - 4_f$.

   (b) Explain how a student who is using the missing addend method could compute $12_f - 4_f$.

   (c) Consider the subtraction problem $31_f - 3_f$. Try using the take-away method to compute this difference in your head. Try using the missing addend method to compute this difference in your head. Which method do you think is easier for this problem? Why?

   (d) Think about the method that you did NOT choose in the previous problem. Provide a subtraction problem that you think would be easier to solve using that method. Why is that method easier for your subtraction problem?

9. (a) Write a Fenland money story that models the difference $402_f - 113_f$.

(b) Use the money to solve your problem. Be sure to make it clear what was done with the money.

(c) Use base fen tiles to compute the difference $402_f - 113_f$.

(d) Use the standard subtraction algorithm to compute the difference $402_f - 113_f$. Show all work.

(e) For **every digit** that shows up in your work for problem 9d, describe where that digit shows up in the base fen tiles above (problem 9c).

10. Max computed $302_f - 104_f$ in the following way.

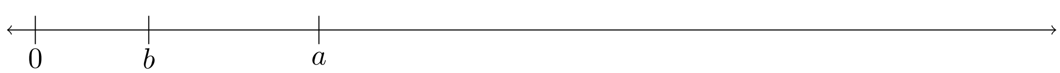
$$\begin{array}{cccc} & 2 & & 12_f \\ & \not{3} & 0 & \not{2}_f \\ - & 1 & 0 & 4_f \\ \hline & 1 & 0 & 3_f \end{array}$$

(a) By analyzing his work, what can you conclude that Max understands?

(b) By analyzing his work, what can you conclude that Max does not understand?

(c) Think of a question or a series of questions that you could ask Max to help him recognize that he made a mistake and to help him to see how to do it correctly. Please record this as a back and forth conversation between you and Max. Your questions could assume that Max has worked with base fen tiles, with money in base fen, or with place value cards, if you wish.

11. The numbers a and b are shown on the number line below. Find the exact location of each of the following numbers.



(a) $a + b$ (b) $3a$ (c) $3a - b$ (d) $2(b + a)$ (e) $a - b$

12. Recall that when using an open number line the goal is to perform jumps that can easily be done in your head. For these to be authentic exercises you must force yourself to work in your head. (In other words, don't figure out the answer in some other way first!)

(a) Use an open number line to compute $212_f + 143_f$.

(b) Use an open number line and the missing addend method to compute $413_f - 124_f$.

(c) Use an open number line and the take away method to compute $413_f - 124_f$.

13. Max knows that the distributive property of multiplication over addition says that $14_f \times 4_f$ is equal to $(3_f \times 4_f) + (11_f \times 4_f)$, but he doesn't really see why they are equal.

    (a) Use arrays to explain to Max **why** the answer to these two problems is the same.

    (b) Give another explanation (that does not use arrays) as to **why** the answer to these two problems is the same.

14. A student is computing $23_f \times 34_f$ and is trying to apply the distributive property. He rewrites 23_f as $20_f + 3_f$ and 34_f as $30_f + 4_f$. He then computes $(20_f \times 30_f) + (3_f \times 4_f)$. How can an array be used to help the student see his mistake?

15. Consider the problem $4_f \times 30000_f$.

    (a) Describe a shortcut that enables you to compute this product using only single digit multiplication.

    (b) Explain why this shortcut works.

16. Consider the problem $4000_f \times 30000_f$.

    (a) Describe a shortcut that enables you to compute this product using only single digit multiplication.

    (b) Explain why this shortcut works.

17. (a) Use an expanded array to compute $134_f \times 23_f$.

    (b) Use the expanded algorithm to compute $134_f \times 23_f$.

    (c) The work for computing $134_f \times 23_f$ using the standard multiplication algorithm is shown below. Explain where the boxed 2 and the boxed 1 show up in the expanded array above (problem 17a).

$$\begin{array}{cccc} & 1_f & 1_f & \\ & 2_f & 2_f & \\ & 1 & 3 & 4_f \\ \times & & 2 & 3_f \\ \hline 1 & 0 & \boxed{1} & 2_f \\ 3 & \boxed{2} & 3 & 0_f \\ \hline 4 & 2 & 4 & 2_f \end{array}$$

    (d) Max is confused because in part 17a one of the products he computed was $4_f \times 20_f$, but he doesn't see the answer to this problem in the standard algorithm. Explain to Max where the answer to $4_f \times 20_f$ shows up in the standard algorithm.

18. (a) Use division by grouping to compute $123_f \div 13_f$. Please be sure to state your answer to the division problem and to show enough work so that it is clear how you arrived at your answer.

    (b) Use division by sharing to compute $123_f \div 13_f$. Please be sure to state your answer to the division problem and to show enough work so that it is clear how you arrived at your answer.

19. Mario has never heard of Fenland but he is observing some students working on a Fenland money problem. He describes what he sees below.

    > A group of eight people are trying to fairly share some Fenland money. They have three pinks, three greens, two yellows, and two blues to divvy up. They first notice that there aren't enough pinks for each to get one, so they trade their three pinks in for fifteen greens, which brings their total number of greens to eighteen. Each person gets two greens, leaving two greens. They turn these two greens in for ten yellows, bring their total number of yellows to twelve. Each person then gets one yellow, which leaves four yellows. They turn these four yellows in for twenty blues, so now they have a total of twenty-two blues. Each person thus gets two blues, leaving six blues that can't be distributed.

    Show the complete long division calculation in **base fen** that corresponds to the above paragraph.

20. Write a description of the process, similar to Exercise 19, that could correspond to the following division problem.

$$
\begin{array}{rccccl}
 & & 2 & 1 & 4_f & r.2_f \\ \cline{2-5}
4_f \,\big) & 1 & 4 & 2 & 3_f & \\
- & 1 & 3_f & & & \\ \cline{2-3}
 & & 1 & 2_f & & \\
- & & & 4_f & & \\ \cline{3-4}
 & & & 3 & 3_f & \\
- & & & 3 & 1_f & \\ \cline{4-5}
 & & & & 2_f &
\end{array}
$$

21. (a) Use the standard long division algorithm to compute $3421_f \div 3_f$. (Be sure to show all steps.)

    (b) Max notices that after you "bring down" the 2, you have 12_f. Max asks, "What are there 12_f of?" Respond to Max's question.

(c) Max then asks, "Why did you only bring down the 2 and not the 21_f?" Respond to Max's question.

22. Max distributed 3 pinks, 4 greens, 2 yellows, and 3 blues evenly among four people in the following way.

- Exchange 2 yellows for 20_f blues.
- Distribute 3 blues to each person.
- Exchange 3 pinks for 30_f greens.
- Distribute 4 greens to each person.
- Exchange 3 greens for 30_f yellows.
- Distribute 3 yellows to each person.
- Exchange 3 yellows for 30_f blues.
- Distribute 4 blues to each person.

Max did not distribute the money in the most efficient way so it will not match up with the standard long division algorithm.

Write down what the long division would look like for the above distribution of money. Be sure to show **all** steps.

3 Back to Base Ten

In the previous section we learned how to add, subtract, multiply, and divide numbers in multiple ways, including the standard algorithms. We explored these ideas in the Fenland system to put us in a less comfortable setting which forced us to think more deeply about what is actually going on. We now return to our beloved number system. All of the activities we did in the previous section can be done in our number system. The ONLY thing that would change is that the exchange rate in Fenland is “fen to one”, where as in our number system the exchange rate is “ten to one”. With the insight gained from working in Fenland, we will now take some time to explore our understanding of arithmetic of whole numbers in our base ten system.

3.1 Re-exploring the System We Know

An extremely powerful activity students in K-6 classrooms can do to improve their success in the mathematics classroom is to perform mental math tasks. These tasks involve a student finding the answer to some numerical problem without use of paper or pencil. These tasks encourage students to visualize numbers in multiple ways choosing the one that serves them best in the moment. Below are some examples. The reader is encouraged to complete these tasks without use of pencil and paper. Remember to be flexible with your numbers!

Question 3.1.1. How could we compute the answer to each of the following problems by working completely in our heads?

$34 + 48$ $\qquad$ 15×11 $\qquad$ 26×9 $\qquad$ 73-45 $\qquad$ $6318 \div 9$

Open number lines are also a way to develop students number sense and flexibility with numbers. Rather than requiring a student to do all work in their head, they are allowed to use an unmarked number line. Recall that the “open” aspect of the number is that the student chooses what numbers to put on there and how to use them. In both mental math problems and in open number line problems, making a ten is an extremely useful tool. We previously mentioned the notion of a jump of ten and a jump via ten which are two valuable strategies students’ can use to make computations easier.

Question 3.1.2. How could we compute the answer to each of the following problems using an open number line?

$289 + 461$ $\qquad$ $228 - 183$ $\qquad$ $267 + 356$ $\qquad$ $513 - 290$

It is interesting to see how many different ways students will come up with the answers to the mental math problems and the open number line problems. The reader is encouraged to come up with as many ways as they can to do each problem. As we have mentioned previously, just because we teach the “standard algorithms” does not mean that is the way we must force our students to do every problem. As teachers, we will often have to look at what a student does and decide if their method is correct or not, so our own ability to find and critique alternate methods is extremely valuable for our success in the classroom as teachers.

In the video below, we will compute $352 - 187$ and $751 - 493$ in a non-standard way. We may want to refer back to this method later, so for lack of a better name, we will call this method "Subtraction by Magic Addition".

Subtraction by Magic Addition Video

Question 3.1.3. If we were to use Subtraction by Magic Addition on the subtraction problem $387 - 254$, what addition problem would we compute?

Question 3.1.4. Is Subtraction by Magic Addition a valid method for computing a difference? In other words, will this method always produce the correct answer to a given subtraction problem?

In the video below, we will compute 376×58 in a non-standard way. This method actually happens to be more standard in other countries and is often referred to as the **lattice algorithm**.

Lattice Algorithm Video

Question 3.1.5. Could the lattice algorithm still produce a correct answer if the numbers were placed differently on the outside of the box and/or the diagonals went a different direction?

In the video below, we will compute $517 - 345$ in a non-standard way. We may want to refer back to this method later, so let's call this method the expanded algorithm for subtraction.

Expanded Subtraction Video

Question 3.1.6. Why is one of the numbers in parenthesis?

Question 3.1.7. Is the expanded algorithm for subtraction a valid method for computing a difference?

3.2 Exercises

1. For each of the arithmetic problems below give two different ways to compute them using mental math. By "mental math" we mean using a method other than the standard algorithm and performing all calculations in our head. Please be sure to describe your method carefully.

   (a) 42×9 (b) $315 - 96$ (c) $251 + 295$

2. Use an open number line to compute the following arithmetic problems. Please make it clear what your method is and that you are getting your answer only from the number line and computations done in your head.

   (a) $376 + 223$ (b) $412 - 278$ (c) $627 - 198$

3. Four children used mental mathematics to compute $753 - 78$. Their notes on how they did it are below. For each child, show the student's method could be displayed using an open number line.

   (a) Twenty-two more gets me to 100, and another 600 then gets me to 700, and then I just need 53 more after that. So $22 + 600 + 53 = 675$.
   (b) Take away 50 to get 703 (so I still need to take away 28). Then take away 3 more to get to 700, and 20 more to get to 680 (so I still need to take away 5). Take the last 5 away, and I stop at 675.
   (c) If I took away 100, that would be too much to take away, but I'd have 653. Since I took away 22 too much, I can add it back now to get $653 + 22 = 675$.
   (d) 700 more gets me to 778, but I added too much. But 778 is 28 more than 750, but I am trying to get to 753, so I just need to add 3 more. I have $700 - 28 + 3 = 700 - 25 = 675$.

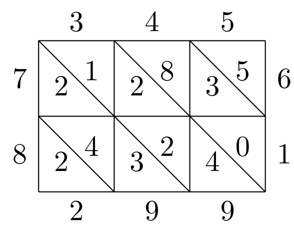
4. Without actually computing the products, use an array and provide a written explanation of which is larger, 37×46 or 36×47.

5. Macy has a method that she calls "Balancing" to help her compute certain sums in her head. For example, if she is trying to compute $39 + 41$ in her head, she will instead compute $40 + 40$. She then knows that since $40 + 40 = 80$, it must also be true that $39 + 41 = 80$.

   (a) If Macy were to use her "Balancing" method on each of the following addition problems, to what would she change each of them? $69 + 71$? $58 + 62$? $46 + 54$?
   (b) Max doesn't quite understand what Macy is doing in her "Balancing" method. Use the problem $27 + 33$ to explain to Max what Macy is doing.
   (c) Why does Macy's method work?
   (d) Max thinks this trick is pretty cool, so he tries to use it for multiplication. He needs to compute 39×41, so he thinks it will be equal to 40×40. Unfortunately this is not correct. Without actually computing either of the products, determine which one is larger and by how much.

6. (a) Use Max's Magic Addition to compute $537 - 269$.
   (b) Explain why Max's Magic Addition works.

7. (a) Use an expanded array to compute 345×78.
   (b) Use the lattice algorithm to compute 345×78.
   (c) In the lattice algorithm look at the three digits in the diagonal between 7 and 1, and indicate where each of those digits show up in the expanded array.
   (d) Max decided to try the lattice algorithm with the diagonals going the other direction. He is computing 345×78 below.



According to Max's lattice, $345 \times 78 = 29,916$, but Max knows that the correct answer is 26,910. Explain to Max why the diagonals going in this direction give him the wrong answer.

8. In algebra students have to learn how to simplify expressions like $3x+5x$ for example. Students sometimes just memorize a rule that tells them that in situations like this you just add the coefficients. Unfortunately that does not allow students to truly understand what is happening. Give a careful explanation as to what property is being used when simplifying $3x + 5x$ to $8x$?

9. (a) Use an expanded array to compute 13×17.
   (b) How could you alter the array above to see how to compute $(x + 3)(x + 7)$? Find this algebraic product using your altered array.
   (c) Use an expanded array to compute 23×47.
   (d) If you altered the above array in a manner similar to Exercise 9b to include x's, what algebraic product would it be computing? Justify your response.

10. Consider the equation below, where a, b, and c are whole numbers.

$$a + b - c = a - c + b$$

   (a) Is this statement always true, sometimes true, or never true? Justify your response.
   (b) Torsten said that this statement cannot always be true because subtraction is not commutative. What is the flaw in Torsten's argument? (Be sure you are addressing the flaw in his argument, not in his conclusion.)

11. (a) State the left distributive property of division over subtraction.
   (b) State the right distributive property of division over subtraction.
   (c) One of the above properties hold and one doesn't. Justify the property that holds.

(d) For the property that does not hold, is the corresponding equation sometimes true or never true?

12. Suppose a, b, and c are whole numbers such that $a < b < c$. If more than an example is needed, please justify using a number line.

    (a) Is the inequality $b - a < c - b$ always true, sometimes true, or never true? Justify your response.

    (b) Is the inequality $b - a < c - a$ always true, sometimes true, or never true? Justify your response.

    (c) Is the inequality $ba < ca$ always true, sometimes true, or never true? Justify your response.

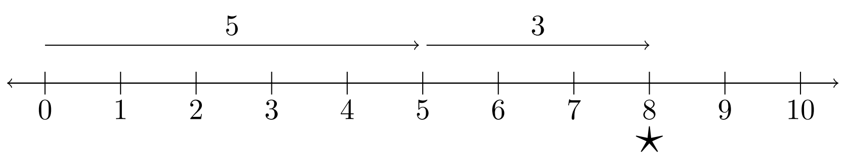
4 Integers

4.1 Definition of Negative Numbers

So far we have been solely working with the set of numbers called, **Whole Numbers**. The set of whole numbers are the numbers $0, 1, 2, 3, 4, 5, 6, \ldots$. This is the first set of numbers students learn in school. As they continue in their academic career they will be introduced to fractions, negative numbers, and irrational numbers. In this section we will focus on the introduction of negative numbers. The set of whole numbers together with their negatives is called the set of **Integers**, and we usually use the symbol $\mathbb{Z}$ to represent the set of integers. So $\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$.

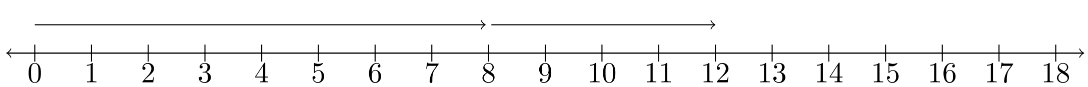
We are going to rely fairly heavily on the use of the number line to develop our understanding of negative numbers, so we are going to take a few moments to solidify our facility with the number line. Recall that a number can be represented on the number line in multiple ways. First as a label on a tick mark and secondly as an arrow. As a refresher, look back at Example 2.4.2.

The arrows are what allowed us to add on the number line. For example, if we wanted to compute $5 + 3$ on the number line, we started at 0, went to 5 (first arrow) then from there went a distance of 3 (second arrow). The visual of this is shown below.



The answer to the addition problem $5 + 3$ can then be found on the number line by looking at the label of the tick mark where the second arrow ended (marked with a star). In this case, that tick mark is 8, so $5 + 3 = 8$.

Example 4.1.1. What addition problem is modeled below? What is the answer to the addition problem?



Possible Solution. The first arrow represents the number 8 and the second arrow represents the number 4, so the addition problem is $8 + 4$. To find the answer to the addition problem, we need to determine the label on the tick mark where the second arrow ended. That tick mark is 12, so $8 + 4 = 12$. □

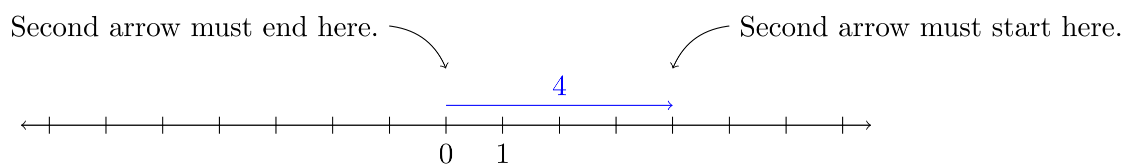
Now that we have refreshed our minds on how to work with number lines, we are ready to tackle negative numbers. As with all things in mathematics, we must start with a definition. In other words, we must first start by defining what that little line in front of the number means. In other words, we must first define what ^{-}a means, and then build all of our understanding up from the definition.

Definition 4.1.2. ^{-}a is the number that when added to a results in 0.

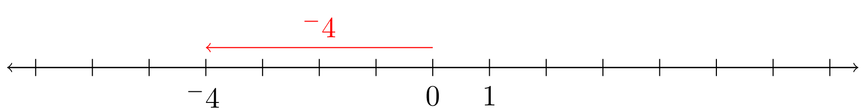
That is the formal definition, but let's make sense of it. The definition is saying that the only thing we know about $^{-}4$, for example, is that $4 + {}^{-}4 = 0$. We will build up our understanding of integers just from this one idea.

We will start with determining where negative numbers are placed on the number line. Keep in mind that every number that we have dealt with so far can be represented by an arrow with a certain length. What do those arrows representing negative numbers look like? Keep in mind we must build up from all of the mathematics we have developed thus far, which means as far as negatives go, we only have the definition.

Let's consider the example of $^{-}4$. Where should $^{-}4$ be placed on the number line. Can we reason out logically where it should be? Well, we know that $4 + {}^{-}4 = 0$, and we know how to represent an addition problem on the number line. Let's see where that gets us. Normally for an addition problem we would draw the arrow representing the first number, which in our case is 4. We would then start at the end of that arrow and draw an arrow representing our second number, but in our case that second number is $^{-}4$ and we don't know what arrow represents that yet. However, we do know the answer to the addition problem is 0, so the second arrow must end at 0. From this information we can conclude what arrow must represent $^{-}4$.

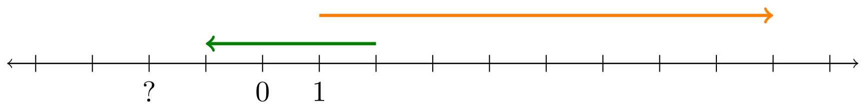


Therefore the number $^{-}4$ is represented by an arrow pointing left with a length of 4. This now allows us to label the correct tick mark with $^{-}4$. Recall that we label a tick mark by using the arrow representing that number and starting at 0. The tick mark the arrow ends on is labeled by that number. We know $^{-}4$ is represented by an arrow of length 4 pointing left. When we place that arrow starting at 0, we can then label the correct tick mark with $^{-}4$. This is shown below.



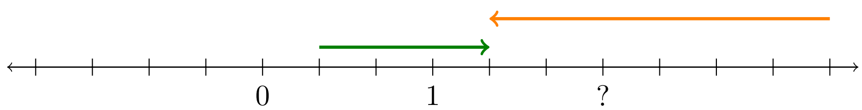
Using this idea for all negative whole numbers, we now see where the negative numbers are located on the number line. The important representation we figured out and that we will use frequently is that any negative number is represented by a left pointing arrow whose length is equal to its corresponding positive number.

Example 4.1.3. What numbers are represented by the arrows below? What number should be placed on the tick mark?



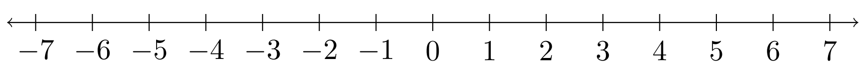
Answer. The green arrow represents the number -3. The orange arrow represents the number 8. The tick mark should be labeled -2. ☐

Example 4.1.4. What numbers are represented by the arrows below? What number should be placed on the tick mark?



Answer. The green arrow represents the number 1. The orange arrow represents the number -2. The tick mark should be labeled 2. ☐

Notice that just from the definition of a negative number and our understanding of addition on the number line, we were able to determine the location of negative numbers on the number line. Note that we did not need to be told where to put the numbers, but that we were actually able to logically determine where they must be. A number line including both positive and negative whole numbers is shown below.

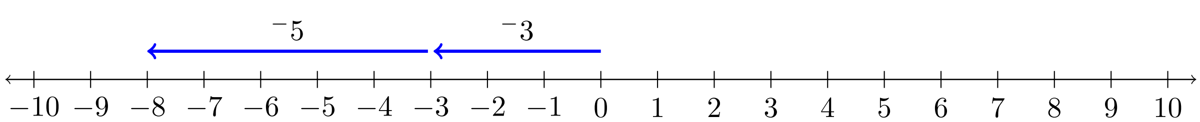


4.2 Addition of Integers

We now want to begin doing arithmetic with negative numbers. Don't forget that we have already defined all of the operations, so we know what $+, -, \times$, and $\div$ means. Therefore, when we operate with negative numbers it should be built from the understanding we already have of the operations. In particular, we don't want to just define a bunch of new rules about how to add negative numbers. Rather, we will build up from what we already know about addition of whole numbers to determine how to add negative numbers.

Example 4.2.1. Use a number line to find the sum $^{-}3 + {}^{-}5$.

Possible Solution. Just as we did with whole numbers, to add on the number line we place an arrow representing the first number (starting at 0) followed by an arrow representing the second number. The answer to the addition problem is where the second arrow ends.



Since the second arrow landed on the tick mark labeled $^{-}8$, then $^{-}3 + {}^{-}5 = {}^{-}8$. ☐

The above example contained an addition problem where both addends were negative. After doing a few more examples where we add two negative numbers we should be able to make some generalizations.

Question 4.2.2. If we add two negative numbers, will the sum always be negative, sometimes (but not always) be negative, or never be negative?

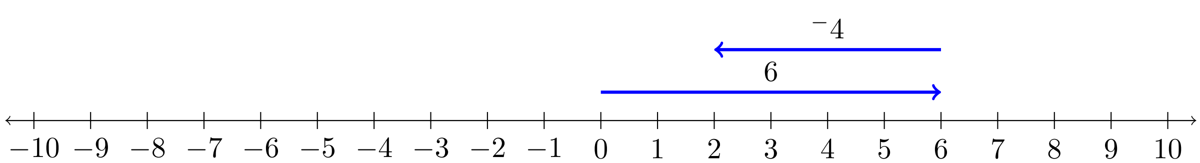
Although the number line is helpful to understand how to add two negatives, we certainly do not want to have to draw a number line every time we are faced with an addition of integers problem. However, if we imagine what the problem would look like on the number line we should be able to determine what the answer will be.

Question 4.2.3. How could we find the sum of two negative numbers without using the number line?

With your answer to the question above, we should be able to find the sum of any two negative numbers without use of a number line. Obviously, we already know how to add two positive numbers, so the only sum left is that of a positive and negative number. Let's take a look at a few examples.

Example 4.2.4. Use the number line to find the sum $6 + {}^{-}4$.

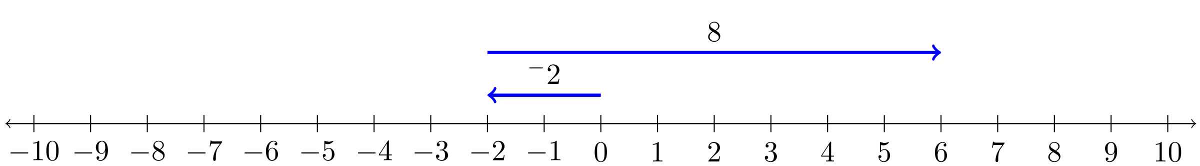
Possible Solution. We again place an arrow representing 6 so that it starts at 0. We then place an arrow representing ${}^{-}4$ at the end of the first arrow. The answer to the addition problem is the location where the second arrow ends.



The second arrow ended at 2, so $6 + {}^{-}4 = 2$. □

Example 4.2.5. Use the number line to find the sum ${}^{-}2 + 8$.

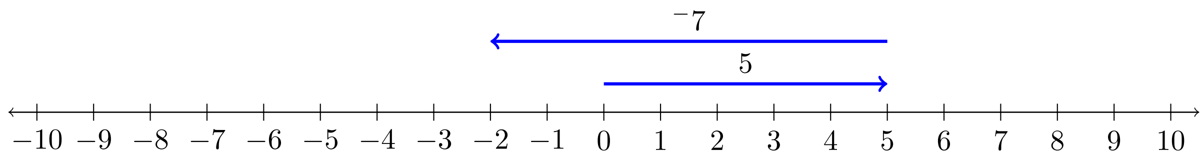
Possible Solution. We will proceed similarly to the above solution.



The second arrow ended at 6, so ${}^{-}2 + 8 = 6$. □

Example 4.2.6. Use the number line to find the sum $5 + {}^{-}7$.

Possible Solution. We will proceed similarly to the above solution.



The second arrow ended at $^{-}2$, so $5 + {}^{-}7 = {}^{-}2$. □

Now that we have gone through a few examples, we should be able to make some generalizations about the sum of a positive and a negative number. In the above examples, we found answers that were positive and answers that were negative.

Question 4.2.7. Without actually computing the sum of a positive and negative number, how can we tell if the answer will be positive or negative?

As before, we want to be able to compute the sum without having to draw a number line. From the question above, we have addressed whether or not the answer will be positive or negative. We now need to address how to actually get the answer. We should be able to imagine the number line so that we can describe how to find the sum without actually needing to draw the number line.

Question 4.2.8. How could you find the sum of a positive and negative number without using a number line?

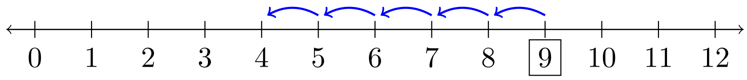
Upon answering, and of course justifying, each of the questions in this section, we understand how to efficiently compute the sum of any two integers. We will proceed, just as we did with whole numbers, to subtraction.

4.3 Subtraction of Integers

As mentioned in the addition section, we do not want to just state a bunch of rules that say how to subtract within the integers. Rather, we want students to see that what they are doing with subtraction in the integers is just an extension of what they did with subtraction of whole numbers. Therefore the goal of this section is to recognize methods that allow us to compute differences within the integers in ways that are reasoned not memorized.

We will again rely heavily on the number line, so let's spend some time revisiting subtraction in that setting. Recall that with subtraction, we could use the take away method or the missing addend method. If two students are using a number line to subtract, the work of a student using the take away method will look different than the work of a student using the missing addend method. The answers, of course, are the same but the method is different.

Let's first look at subtraction on the number line via the take away method. Consider the subtraction problem $9 - 5$. If we are thinking of this as 9 take away 5, then on the number line we would mark 9 and go back 5 units. The answer to the subtraction problem is given by the spot on the number line where we end up. We show this below.



Since we ended up at 4, then $9 - 5 = 4$.

Let's see how far this can get us with integers. Below we have eight different subtraction problems. Try to compute each of these differences on a number line using the take away method in the same way we did with whole numbers.

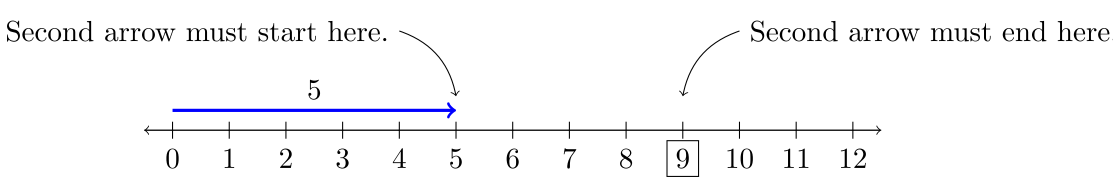
$5 - 3$ $4 - 7$ $^{-}9 - {}^{-}5$ $^{-}3 - {}^{-}8$

$^{-}6 - 2$ $2 - {}^{-}3$ $9 - {}^{-}3$ $^{-}5 - 8$

Question 4.3.1. For which of the problems above were you able to use the method just as in whole numbers? What do all of those problems have in common?

One of the problems from above that cannot be done on the number line using the take away method is $2 - {}^{-}3$. If we were to use take away on this problem, then we would need to be thinking about "take away $^{-}3$" on the number line. We understand how to do "take away 3" on the number line for example, but we do not understand "take away $^{-}3$" on the number line. Perhaps the missing addend method will work better for us in this case. Let's review how the missing addend method works on the number line.

Consider the difference, $9 - 5$. To compute this using the missing addend method we need to determine what to add to 5 in order to get 9. In other words, $5 + ? = 9$. To figure out the question mark using the number line we need to rely on our understanding of addition on the number line. We will draw an arrow starting at 0 that represents 5. We then need to draw a second arrow, but we don't know how to draw that arrow. But wait...we know the answer to the addition problem, so we know that second arrow must end at 9. We can now determine what the arrow must be and thus discover what the question mark is. A picture summarizing this idea is shown below.

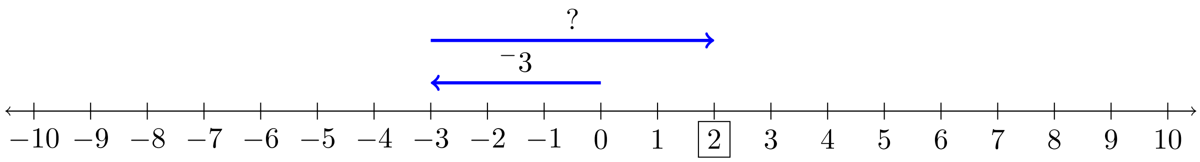


Therefore the second arrow must be pointing right and have a length of 4. Thus $9 - 5 = 4$.

To summarize then, when we are computing $a - b$ using the missing addend method on the number line, we are asking ourselves, what arrow do we need to get us from b to a on the number line. The number that arrow represents is the answer to our subtraction problem. We will employ this method in the following example.

Example 4.3.2. Use the number line to compute $2 - {}^{-}3$.

Possible Solution. To compute $2 - {}^{-}3$, we need to solve ${}^{-}3 + ? = 2$. In other words, on the number line, how do we get from ${}^{-}3$ to 2?

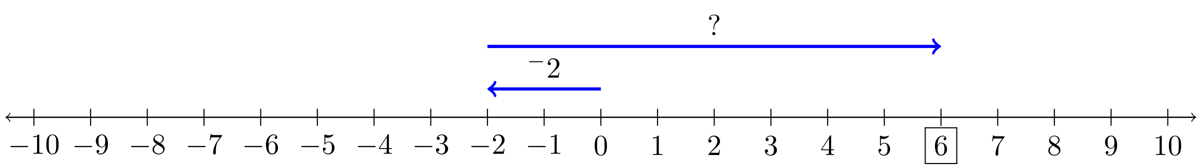


The question mark arrow points to the right and is 5 units long. Therefore the question mark arrow represents the number 5. Thus $2 - {}^{-}3 = 5$. □

Yes indeed! The missing addend method allowed us to determine the difference when the number we were subtracting was negative. It is worth noting that even the problems above that we said could be done via take away can also be solved using the missing addend method. For example, $4 - 7$ was a problem above that could have been done via take away, but it can also be done via missing addend. Let's take a look at a few more examples.

Example 4.3.3. Use the number line to compute $6 - {}^{-}2$.

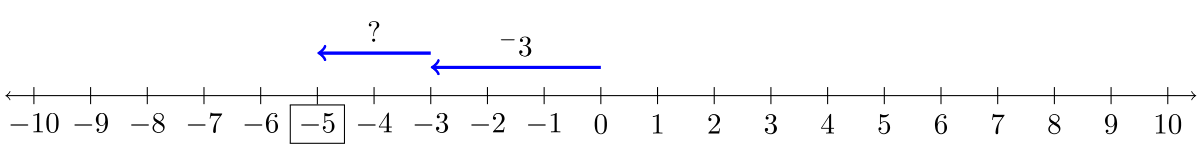
Possible Solution. To compute $6 - {}^{-}2$, we need to solve ${}^{-}2 + ? = 6$. In other words, on the number line, how do we get from ${}^{-}2$ to 6?



The question mark arrow points to the right and is 4 units long. Therefore the question mark arrow represents the number 4. Thus $6 - {}^{-}2 = 8$. □

Example 4.3.4. Use the number line to compute ${}^{-}5 - {}^{-}3$.

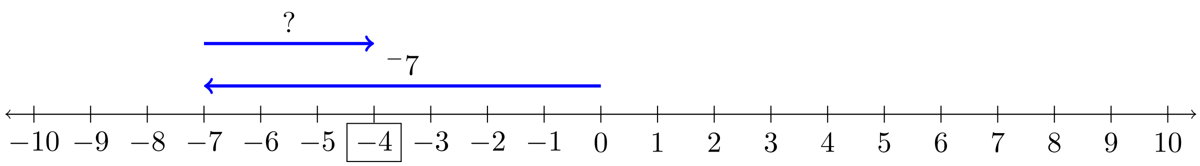
Possible Solution. To compute ${}^{-}5 - {}^{-}3$, we need to solve ${}^{-}3 + ? = {}^{-}5$. In other words, on the number line, how do we get from ${}^{-}3$ to ${}^{-}5$?



The question mark arrow points to the left and is 2 units long. Therefore the question mark arrow represents the number ${}^{-}2$. Thus ${}^{-}5 - {}^{-}3 = {}^{-}2$. □

Example 4.3.5. Use the number line to compute ${}^{-}7 - {}^{-}4$.

Possible Solution. To compute $^{-}4 - {}^{-}7$, we need to solve $^{-}7 + ? = {}^{-}4$. In other words, on the number line, how do we get from $^{-}7$ to $^{-}4$?



The question mark arrow points to the right and is 3 units long. Therefore the question mark arrow represents the number 3. Thus $^{-}4 - {}^{-}7 = 3$. ☐

Through these examples, we have developed some facility with subtraction of integers. We of course do not want to have to use number line forever however. Let's try to use our knowledge of the number line, but not actually draw a number line, to deepen our understanding of subtraction of integers.

Consider the following problems.

$29 - 55$ $\quad$ $^{-}38 - 21$ $\quad$ $^{-}45 - {}^{-}52$ $\quad$ $32 - {}^{-}58$

Question 4.3.6. Without actually computing any of the answers, how can you determine whether the answer will be positive or negative?

Question 4.3.7. How could you compute the answer to each of the problems without having to draw a number line?

The purpose of the previous question is to solidify the idea that we have developed a method for computing the difference of any two integers without use of any memorized rules. Therefore the method used to arrive at the answers should come from our work on the number line, and thus the arithmetic we chose to do can be justified.

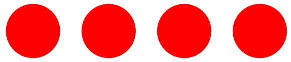
4.4 An Alternate Model for Integers

The strength of the number line to encourage reasoning and understanding of negative numbers cannot be overstated, so this section is not in place to suggest that the counters described below could be used in place of number lines. Rather, they allow students a second look at integers from a different viewpoint. To model the integers we will use colored counters, which are small disks that have two different colors (or perhaps a + and −) on them. In our class the counters are red on one side and yellow on the other. Just so we can easily communicate, we will agree to use the red side to represent negatives and the yellow side to represent positives.

Example 4.4.1. What is the value of the counters below?

Answer. We have 5 yellow counters, and we agreed yellow is positive. Therefore the value of these counters is 5. $\square$

Example 4.4.2. Give a collection of counters whose value is $^{-}4$.

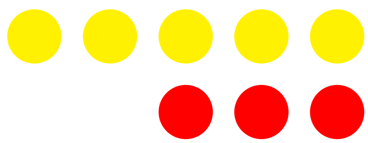
Possible Solution. 

$\square$

Question 4.4.3. Are there other collections of counters that would have a value of $^{-}4$?

In both of the above examples, we only used one color. What happens if we have both colors showing up? For example, what if we have one red and one yellow counter. What is the value of that collection? To answer that we need to go back to our definition of negatives and our understanding of addition as all together. We can think of one yellow counter and one red counter as $1+{}^{-}1$. According to the definition of a negative number, we know that $1+{}^{-}1$ equals 0. Therefore the value of one yellow counter and one red counter is 0. When using the counters we often refer to one red counter and one yellow counter as a **zero pair**.

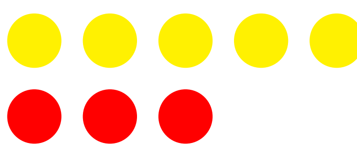
Example 4.4.4. What is the value of the counters below?



Answer. This collection of counters has a value of 2.

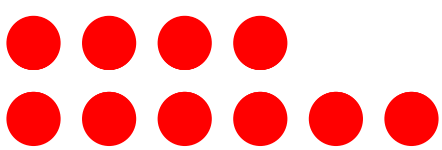
Example 4.4.5. Use counters to compute $5 + {}^{-}3$.

We place 5 yellow counters and 3 red counters down and put them all together. Now we just need to figure out the zero pairs, to see what the value of the counters are. The visual is shown below.



Therefore $5 + {}^{-}3 = 2$. $\square$

Example 4.4.6. Use counters to compute ${}^{-}4 + {}^{-}6$.

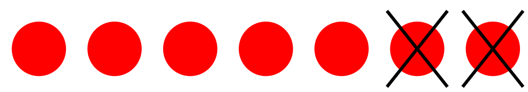


All together we end up with 10 red counters. Thus ${}^{-}4 + {}^{-}6 = {}^{-}10$. $\square$

Let's now move on to subtraction problems. Recall that when we were subtracting on the number line, there were certain problems that we were not able to do via the take away method. With the counters, we are able to do either take away or missing addend. This may seem like a reason to favor counters over number line, but don't be too quick to make that decision. I will address this at the end of the section.

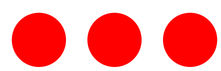
Example 4.4.7. Use colored counters and the take away model to compute $^{-}7 - {}^{-}2$.

Possible Solution. We begin by laying down 7 red counters to represent $^{-}7$. We then need to take away a value of $^{-}2$. That means we need to take away 2 red counters.



We are left with 5 reds, so $^{-}7 - {}^{-}2 = {}^{-}5$. □

Let's now consider the problem $^{-}3 - 4$. Suppose we want to compute this difference using colored counters and the take away method. To use the take away method we must be thinking $^{-}3$ take away 4. Therefore we need to lay down 3 red counters to represent $^{-}3$.



We now need to take away a value of 4. In other words, we need to remove 4 yellow counters. And there's the problem! We don't have 4 yellow counters to take away. What to do, what to do... If only we could represent $^{-}3$ in a different way. In particular, in a way that would allow us to remove 4 yellow counters.

Question 4.4.8. How could we represent $^{-}3$ so that we could perform "take away 4"?

After answering the question above and taking away the 4 yellow counters, we will be left with 7 red counters. Therefore $^{-}3 - 4 = {}^{-}7$.

Although, we certainly can use this process to subtract any two numbers using colored counters and take away, it is pretty clear that it is not quite as intuitive as the way we could have computed this on the number line. In particular, on the number line a student computing $^{-}3 - 4$ using take away will quickly see that this problem is equivalent to $^{-}3 + {}^{-}4$. However, using counters $^{-}3 - 4$ looks very different from $^{-}3 + {}^{-}4$, which would involve laying down 3 reds and then 4 reds. In other words, students using the number line will most likely recognize the relationship between adding a negative and subtracting its positive. Where as this connection is not as clear with the counters. (That is not to say that it cannot be justified using counters, just that it is not quite as intuitive.)

Recall that on the number line there were problems that were easier to do using missing addend, so maybe that is the case with counters as well. Let's revisit the problem $^{-}3 - 4$ using the missing addend method instead. We would need to be thinking about the equation $4 + ? = {}^{-}3$. In other words, what would we have to add to 4 to end up with $^{-}3$. In the language of counters, we would need to represent 4 by laying down 4 yellow counters. We then need to determine what counters we need *add* to the collection so that the value is $^{-}3$.

We begin by laying down 4 yellow counters.

Question 4.4.9. What color counters will we want to add to this collection?

Question 4.4.10. How many counters will we need to add in order to reach a value of $^{-}3$?

In answering the two questions above, you should be able to see how $^{-}3 - 4$ is in fact equal to $^{-}3 + {}^{-}4$. In addition, we can see how this would generalize if the numbers change. Notice that there is a connection to how we saw this fact using the number line and how we see this fact with the counters.

This brings us back to the discussion regarding which method, number line or counters, is "better". They both have their value, and if given the time, letting students experience both would be wonderful. However, if we only have time to use one method, then for the long term benefit my personal feeling is that the number line is the better choice. If students are working on a number line with integers, then this is just an extension of how they already worked with whole numbers. Therefore the students are much less likely to think of negative numbers as some weird being, but rather a new set of numbers that they can work with just a before. How students feel about negative numbers is important, and the normalcy of the number line is a definite benefit here.

4.5 An Integer Interlude

In the previous sections, our main goal was to develop facility with addition and subtraction of integers. We were more concerned with students intuitively determining the answers rather than memorizing some rules. However, in light of the fact that the students will be moving on to algebra, we cannot deny the fact that they will need some properties at their disposal. These properties should be grounded in what they have done with their intuitive work computing in the integers.

One of the properties we have already mentioned that is important for students to have at their disposal moving forward is the connection between adding a negative and subtracting its positive. This property is stated precisely below.

Property 4.5.1. For any numbers a and b, we have that $a + {}^{-}b = a - b$.

In the addition and subtraction sections we used the number line to do our computations. In this setting we can easily justify the above property. In fact, students will probably come up with this property after doing enough examples. It is amazing how clever students can be when they are trying to avoid doing lots of work. We can use this to our advantage by giving them carefully chosen examples that encourage them to find shortcuts. The shortcuts they find are often exactly the properties we want them to learn.

A second property that students will most likely stumble upon with well chosen examples is the connection between subtracting a negative and adding its positive. The property is stated precisely below.

Property 4.5.2. For any numbers a and b, we have that $a - {}^{-}b = a + b$.

Again, our work in the subtraction section should allow us to justify that this property holds. For example, look back at our work for $6 - {}^{-}2$ in Example 4.3.5.

Question 4.5.3. Where does $6 + 2$ appear in that work?

Similarly we should be able to recognize where $4 + 3$ is in our work for $4 - {}^{-}3$. In fact any problem where we are subtracting a negative number from a positive number, we will be able to easily indicate where in our number line work for $a - {}^{-}b$ the $a + b$ appears.

Interestingly though, it is more difficult to see where that $a + b$ appears if a is in fact a negative number. At first glance, the properties above may appear to imply that a is a positive number, but note that it says a can be any number. Therefore a could be negative. For example, the property applies to the subtraction problem ${}^{-}5 - {}^{-}3$. According to the property ${}^{-}5 - {}^{-}3$ is equal to ${}^{-}5 + 3$. If we were to complete the number line work for ${}^{-}5 - {}^{-}3$, it would not be as clear where ${}^{-}5 + 3$ appears, but it is there.

Question 4.5.4. Where does ${}^{-}5 + 3$ appear in the number line work for ${}^{-}5 - {}^{-}3$?

Recall that in the previous discussion we mentioned that with well chosen examples, students could come up with this property. In this case, those *well-chosen examples* will be differences where the first number is positive. That way students have a better chance of developing this property and being able to justify it. Once they can justify it in that setting, as a teacher we will need to decide if we want to push them to justify it in all settings. Or is their understanding in that setting enough to allow them to successfully use the property in all settings. There is no blanket right answer to this decision because it will depend on the current classroom of students being taught.

For our purposes as teachers however, we should be comfortable justifying it in all settings. The number line work is a perfectly acceptable means to justify Property 4.5.4. However, let's also explore a method that does not involve any visuals. Rather, we will give an algebraic proof that $a - {}^{-}b$ is equal to $a + b$. To do this, let's start with the expression $a - {}^{-}b$ and try to work with it in a way that allows us to see that it is in fact equal to $a + b$.

Let's say $a - {}^{-}b = ?$, and let's try to figure out to what else the question mark is equal. Since $a - {}^{-}b = ?$, then that means ${}^{-}b + ? = a$. We need to figure out what to add to ${}^{-}b$ to end up with a. Well, let's first add in b, then so far we have $b + {}^{-}b$. Well, that doesn't equal a..., but wait that equals 0. So we need to add in an a as well, then ${}^{-}b + b + a$ does in fact equal a. Therefore we have figured out what needs to be added to ${}^{-}b$ in order to get a. That thing we added was $b + a$ or equivalently $a + b$. Therefore the question mark is also equal to $a + b$, so $a - {}^{-}b = a + b$.

To be clear, this is a rather sophisticated argument and not necessarily one that would be done with students. However, it is not really that different than how we saw $a + b$ in the number line work.

Question 4.5.5. What connections do the algebraic proof have with the work on the number line?

Let's now take our interlude away from operations and just look at the numbers themselves. Notice that we have not yet addressed what ${}^{-}\,{}^{-}5$ means for example. In other words, what is negative negative 5. We must work with the definition to understand this. Recall that by definition ${}^{-}\bigstar$ is the number that we need to add to $\bigstar$ in order to get 0.

Question 4.5.6. In the case of ${}^{-}\,{}^{-}5$, what is playing the role of $\bigstar$?

Question 4.5.7. How can we use the definition of negative to determine what ${}^{-}\,{}^{-}5$ equals?

This is a good time to bring in some vocabulary. Before doing so, let's think about the development of the number system we are embarking upon. We will do this from an algebraic standpoint. Consider the equation $4 + x = 9$. Notice that to write this equation down we only needed to use whole numbers. But is the solution to this equation a whole number? Yes, the solution is $x = 5$. On the other hand, consider the equation $4 + x = 3$. Again this equation can be written down using only whole numbers, but is the solution a whole number? No, the solution is $x = -1$. This highlights the need for negative numbers. Since there are equations we could not solve with just whole numbers, we need more numbers.

The definition of negatives is exactly this algebraic viewpoint. For example, -5 is defined to be the number that when added to 5 results in 0. In other words, ${}^{-}5$ is the solution to the equation $5 + x = 0$.

Finally we come to the vocabulary. We know that when we add 0 to any number, we get the *identical* number back. Because of this we say that 0 is the **additive identity**. In other words, adding 0 to a value does not change the value. Similarly we can talk about an identity for the operation of multiplication, in other words a **multiplicative identity**. Is there a number that when multiplying a value by that number the value does not change? That was a wordy question, but once it is deciphered we see that yes of course there is! The number one! If we multiplying a value by 1 the value does not change. Thus 1 is the multiplicative identity.

Algebraically speaking, we need to "undo" things in order to solve equations. But what do we really mean by undo? If we think about it carefully we are creating a situation where we end up with an identity of some sort. When we find a number that when adding creates the additive identity, we call that the **additive inverse**. For example, consider the number 8. We know that if we add $^{-}8$ to 8 we get 0 (the additive identity). Therefore $^{-}8$ is the additive inverse of 8. Turn that around and we see that if we add 8 to $^{-}8$ we get 0, so 8 is the additive inverse of $^{-}8$.

This language allows us to discuss weird numbers like $^{-}\,^{-}5$. Is this a negative number or not? On the one hand it does have the negative symbol in front of it. But on the other hand, we know this is equal to 5, which is definitely not a negative number. Therefore just having the symbol in front does not necessarily mean the number is negative. So more precisely we could say that $^{-}\,^{-}5$ is the additive inverse of $^{-}5$. In many textbooks, the word "opposite" is used in place of additive inverse. So for example a textbook will say $^{-}5$ is the opposite of 5. Although this wording is more friendly and makes sense considering the location of numbers on the number line, we do want to make sure students don't lose sight of the actual definition of a negative number.

What about a multiplicative inverse we might ask? Similar to our description above, when we find a number that when multiplying creates the multiplicative identity we call that the **multiplicative inverse**. For example, what can we multiply 2 by in order to get 1 (the multiplicative identity)? Well, $\frac{1}{2}$ times 2 equals 1, so $\frac{1}{2}$ is the multiplicative inverse of 2. But for now we are going to stick with integers and will address fractions later in the book. Notice though that we have described yet another equation that can be written in the whole numbers but whose solution is not a whole number. Namely, the equation $2x = 1$.

Whew! That was a lot of vocabulary, so let's move on a bit. The next order of business is the choice of symbol we use to designate a negative number. We use the same symbol for negative as we do for subtraction. Why is that? Way back in history, did mathematicians just run out of symbols? I think not! So then what is up with using the same symbol? Well, we have already seen a bit of the reasoning behind this in Property 4.5.1. Namely the fact that adding a negative number is the same as subtracting its positive.

We can also see the logic behind this choice in considering the problem $0-3$. What is the answer to this problem? $^{-}3$ of course. Therefore any negative number can be viewed as 0 minus its positive. For example, when we see $^{-}5$ we can think of it as $0 - 5$. Notice that this falls in line with our desire to think about numbers flexibly. Sometimes we may want to think of $^{-}5$ via the definition of a negative, but sometimes we may want to think of $^{-}5$ as $0 - 5$. (You'll be asked to refer back to this paragraph in a later section.)

Okay, one last little trip on our integer interlude. In this section we discussed some properties (or rules) that students will eventually be expected to have at their disposal. Unfortunately this sometimes encourages students to just memorize the rule rather than reason and understand it. In particular, a statement often uttered by students is that "two negatives make a positive". For example, they may cite this to explain that $^{-}\,^{-}5$ is equal to 5. Or perhaps when they are multiplying two negatives (which we will discuss in the next section). However, this statement is misleading. It is not true that everytime we put two negatives together we get a positive. For example, when we add two negatives we get a negative, not a positive. Students also cite this explanation when justifying $a - {}^{-}b = a + b$. Technically that argument is not valid because we do not have two negatives. We have a subtraction sign and a negative. Yes, we know there is a very close connection between the subtraction symbol and the negative symbol as discussed above, but according to the definitions they are not the same thing. We want to be sure we are precise with our language. Sloppy language often leads to misuse of rules.

4.6 Multiplication and Division of Integers

As with both addition and subtraction of integers, we do not want to just state a bunch of rules that say how to multiply integers. Rather, we want students to build on their understanding of multiplication and division of whole numbers to create an understanding within the integers. Once again, our goal for this section is to develop methods that allow us to compute products and quotients within the integers in ways that are reasoned not memorized.

Let's take a moment to recall the definition of multiplication. We defined $a \times b$ to be the value of a groups where the value in each group is b. In other words, 4×3 means the value of 4 groups if there are 3 in each group. Recall that this is equivalent to saying that $4 \times 3 = 3 + 3 + 3 + 3$. In addition to the definition of multiplication, we also know that multiplication is commutative. We also know how multiplication interacts with addition and subtraction. In particular, multiplication distributes over both addition and subtraction. The definition of multiplication and the properties mentioned above will allow us to multiply any two integers.

As usual, we will proceed through a series of examples to achieve our goal.

Example 4.6.1. Compute $4 \times {}^{-}5$.

Possible Solution. According to the definition, $4 \times {}^{-}5$ is the value of 4 groups if each group has a value of $^{-}5$. Therefore $4 \times {}^{-}5 = {}^{-}5 + {}^{-}5 + {}^{-}5 + {}^{-}5 = {}^{-}20$. □

Using this same strategy, the definition of multiplication will allow us to compute lots of products. For example, $3 \times {}^{-}8$, $5 \times {}^{-}7$, and many more. Unfortunately, the definition alone does not allow us to compute products of *any* two integers though.

Question 4.6.2. Why can't we use the definition of multiplication to compute $^{-}4 \times 5$?

Question 4.6.3. What property of multiplication could we apply, so that we would then be able to compute $^{-}4 \times 5$?

After answering the above questions we are at the point where we could find the product of a positive and a negative number in either order. Alas! We have still not reached our goal of being able to multiply *any* two integers. Consider the problem $^{-}4 \times {}^{-}5$.

Question 4.6.4. Why doesn't the answer to Question 4.6.3 allow us compute $^{-}4 \times {}^{-}5$.

It looks like we are on a hunt for other things we have learned about negatives and multiplication that will help us in our endeavor to compute the product $^{-}4 \times {}^{-}5$.* There are properties of multiplication mentioned above that we have not yet used, namely the distributive properties. Moreover, a glance at the second to last paragraph in Section 4.5 may prove useful. (Yes, I know I am sending you on a scavenger hunt now, but it is valuable to look back every once in a while!)

Question 4.6.5. How can we compute $^{-}4 \times {}^{-}5$?

*Note: The method suggested here is certainly not the only way to reason out what a negative times a negative equals. You are encouraged to explore other ways to achieve the same goal.

In answering that final question, we are now able to compute the product of any two integers. As before, we want students to internalize the method and then notice the shortcuts that appear. After enough well chosen examples, students will come up with the rules for negatives and positives. Through the examples they will see that a negative times a positive is negative and that a negative times a negative is positive. To reiterate, students do not need to be told the rule of signs. They can figure it out themselves!

Students will most likely remember the rule of signs in words. Namely, a negative times a positive is negative, etc. Recalling the rule of signs in this way is perfectly fine when students are working with numbers. When they get to algebra however they will need to apply the rule of signs in that setting, so it will be helpful for them to see it written formally at some point. We state the rule of signs formally below.

Property 4.6.6 (Rule of Signs for Multiplication)**.** The following equations hold for any numbers a and b.

$$^{-}a \times b = {}^{-}(a \times b) \qquad a \times {}^{-}b = {}^{-}(a \times b) \qquad {}^{-}a \times {}^{-}b = a \times b$$

We have spent a great deal of time understanding how to multiply integers. Since division is defined in terms of multiplication, we should be able to use all of that hard work we did for multiplication to determine how division with negatives works. Recall that the definition of division says that $a \div b = ?$ means $b \times ? = a$.

Example 4.6.7. Compute $^{-}32 \div 8$.

Possible Solution. According to the definition of division, $^{-}32 \div 8 = ?$ means $8 \times ? = {}^{-}32$. Therefore we need to find a number that when we multiply by 8 we get $^{-}32$. We have a positive number, 8, and need to multiply it by something to get a negative number. Therefore the number we multiply by must be negative. Moreover we know $8 \times 4 = 32$. Therefore $8 \times {}^{-}4 = {}^{-}32$, so $^{-}32 \div 8 = {}^{-}4$. □

In the previous example we saw that in that case a negative divided by a positive was negative. In fact, you may recall that the rule of signs for division is exactly the same as that for multiplication. For example, a negative divided by a positive is always negative. The rule is stated formally below.

Property 4.6.8 (Rule of Signs for Divsion)**.** The following equations hold for any numbers a and b.

$$^{-}a \div b = {}^{-}(a \div b) \qquad a \div {}^{-}b = {}^{-}(a \div b) \qquad {}^{-}a \div {}^{-}b = a \div b$$

The fact that the two rules are identical, barring the change of $\times$ to $\div$, may lead us into a false sense of security about the reasoning behind the rule of signs for division.

Question 4.6.9. Why is the argument below not valid? And what would be a valid argument for why a negative divided by a negative is positive?

> A negative divided by a negative is positive because
> a negative multiplied by a negative is positive.

The previous question only addressed one of the rule of signs for division. We should use the same type of logic to determine the reasoning behind the other two rules of signs for division as well. This is addressed in the exercises.

4.7 Exercises

1. Explain how to use a number line to compute each of the following problems.

   (a) $^{-}3 + {}^{-}8$
   (b) $3 + {}^{-}8$
   (c) $^{-}3 + 8$
   (d) $^{-}3 - 8$
   (e) $3 - {}^{-}8$
   (f) $^{-}3 - {}^{-}8$

2. (a) Use the definition of negative numbers to justify that $^{-}(^{-}5)$ equals 5.
   (b) Use the fact that $0 - a = {}^{-}a$ to justify that $^{-}(^{-}5)$ equals 5.
   (c) Max and Macy are just starting to learn algebra. They see the expression ^{-}x. Macy says that ^{-}x is a negative number, but Max says that it depends. Who is right and why?

3. In Chapter 3 Exercise 10, you were asked to explain why $a - b + c$ will always equal $a + c - b$. At that time we did not have the use of negative numbers, as we do now. Provide a new justification for this fact using your knowledge of negative numbers.

4. In Chapter 3 Exercise 11b you explored some inequalities for whole numbers a, b, and c where $a < b < c$. Let's now assume a, b, and c are integers where $a < b < c$.

   (a) Is the inequality $b - a < c - a$ always true, sometimes true, or never true? Justify your response.
   (b) Is the inequality $ba < ca$ always true, sometimes true, or never true? Justify your response.

5. Consider the problem $^{-}256 + 328$.

   (a) Without actually finding the answer, use the number line to explain how you know the answer will be positive.
   (b) Without actually finding the answer, use colored counters to explain how you know the answer will be positive.

6. Consider the problem $^{-}45 - {}^{-}28$.
   (You are not allowed to use the fact that $a - {}^{-}b = a + b$ in this problem.)

   (a) Without actually finding the answer, use the number line to explain how you know the answer will be negative.
   (b) Without actually finding the answer, use colored counters to explain how you know the answer will be negative.

7. Imagine you are going to use the number line to compute $^{-}256 + 163$.

(a) Without computing the answer, explain how you know the answer is going to be negative. (Remember you are using the number line, so your answer should refer to the arrows on the number line.)

(b) From above, we know that the end of the second arrow will be to the left of zero. Explain how you can figure out how far away from zero the end of the arrow will be without counting.

8. Compute the following sums numerically (i.e. no counters or number lines) in a way that only requires whole number arithmetic. To help with clarity, please provide a written explanation to accompany your computations whenever necessary.

(a) $28 + {}^{-}15$
(b) $23 + {}^{-}34$
(c) ${}^{-}32 + {}^{-}45$
(d) ${}^{-}48 - {}^{-}23$
(e) ${}^{-}25 - 36$
(f) $28 - 51$

9. For each of the computations you made in Exercise 8, use a number line to justify why your method is valid.

10. (a) Use colored counters and the take-away method to compute $5 - {}^{-}4$. (Be sure to make it clear how you arrived at your answer.)

(b) Use the number line and the missing addend method to compute $5 - {}^{-}4$. (Be sure to make it clear how you arrived at your answer.)

(c) Max is comfortable with either of the above methods. Use the example $35 - {}^{-}24$ to help Max understand why the answer to $a - {}^{-}b$ will always equal the answer to $a + b$ for any whole numbers a and b.

(d) What if a is negative in the above question. For example, consider ${}^{-}35 - {}^{-}24$. Use this example to help Max see that $a - {}^{-}b$ still will always equal $a + b$ even if a is not positive.

11. (a) Use the definition of multiplication to compute $4 \times {}^{-}7$. Please show all work.

(b) Explain why ${}^{-}4 \times 7$ cannot be computed using only the definition of multiplication.

(c) What property of multiplication fixes the issue in part 11b and allows us to compute ${}^{-}4 \times 7$? Compute the product, showing all work.

(d) The product, ${}^{-}4 \times {}^{-}7$, also cannot be computed using only the definition of multiplication. Explain why the "fix" you found in part 11c does *not* fix the problem here.

(e) Max is trying to figure out what ${}^{-}4 \times {}^{-}7$ equals, but he is stuck. He asks a fellow classmate for help and she says, "Just use the fact that ${}^{-}7$ is equal to $0 - 7$." How does this fact allow Max to determine what ${}^{-}4 \times {}^{-}7$ equals?

(f) Stacee is trying to figure out what ${}^{-}4 \times {}^{-}7$ equals, so she decides to Google it. She comes across a method that says to start with an expression like ${}^{-}4 \times ({}^{-}7 + 9)$. Google then says that by the distributive property we know that ${}^{-}4 \times ({}^{-}7 + 9) = ({}^{-}4 \times {}^{-}7) + ({}^{-}4 \times 9)$. How can we use that equation to determine what ${}^{-}4 \times {}^{-}7$ equals?

(g) Would the method Stacee found work if we started with the expression ${}^{-}4 \times ({}^{-}7 + 7)$ instead? Justify your response.

(h) Would the method Stacee found work if we started with the expression $^{-}4 \times (^{-}7 + 5)$ instead? Justify your response.

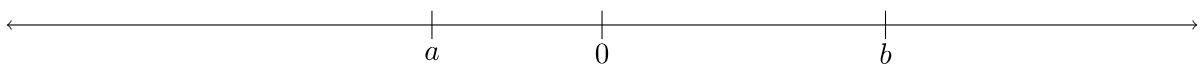
12. Max now knows how to multiply any two integers.

    (a) Use the definition of division to compute $^{-}32 \div 4$. Please make it clear how you arrived at your answer.

    (b) Use the definition of division to compute $^{-}32 \div {}^{-}4$. Please make it clear how you arrived at your answer.

    (c) Explain to Max why a negative divided by a negative is positive.

13. The numbers a and b are shown on the number line below. Find the exact location of each of the following numbers.



(a) $a + b$	(c) $b - a$	(e) $b - 2a$	(g) $\|a - b\|$	(i) $-\|b\|$
(b) $a - b$	(d) $2a + b$	(f) $\|a\|$	(h) $\|b - a\|$	(j) $\|a\| - \|b\|$

5 Exponents (Part I)

5.1 Definition of Exponents

Not to sound like a broken record, but guess where we are going to start? Yes, you guessed it...with a definition. What on earth does it mean when you write a number as a superscript on another number? That is probably exactly what a student is thinking if they have never been introduced to exponents. So let's begin with the definition of exponent and reason our way through the rest of the exponents topic.

Definition 5.1.1. For any natural number n and any number a, we define a^n to mean $a \cdot a \cdots a$, where there are n occurrences of a.

Example 5.1.2. According to the definition above, what does 3^4 equal?

Answer. $3^4 = 3 \times 3 \times 3 \times 3$ □

Of course we do in fact know how to multiply, so we can finish off the problem above: $3^4 = 3 \times 3 \times 3 \times 3 = 81$. As we proceed through examples, keep in mind that we only have the definition of exponents to work with. Before trying to develop any rules we want to be sure we are grounded in the definition.

Example 5.1.3. Write the expression $5 \times 5 \times 5$ using exponents.

Possible Solution. $5 \times 5 \times 5 = 5^3$ □

The solution given above is probably the most common one, but the way the question was worded would make the answer $5^1 \times 5^1 \times 5^1$ a perfectly good answer as well. It is worth noting that sometimes the exponent of 1 throws students for a moment, so we want to be sure we include that in the examples we choose for students.

Example 5.1.4. Write the expression $5 \times 5 \times 3 \times 4 \times 5 \times 3 \times 3 \times 5$ using exponents.

Possible Solution. We first use the commutative property of multiplication to rewrite the given expression.

$$5 \times 5 \times 3 \times 4 \times 5 \times 3 \times 3 \times 5 = 5 \times 5 \times 5 \times 5 \times 3 \times 3 \times 3 \times 4$$

We can now use the definition of exponents to get $5^4 \times 3^3 \times 4^1$. □

We want students to be comfortable going back and forth between the factored form of an expression and an exponent form of the expression. By **factored form of an expression**, we mean an expression written out as a product of numbers (or variables) without exponents. As we saw in previous examples, some expressions have multiple ways to write it using exponents. By **exponent form of an expression** we mean an expression in which the use of exponents is employed. For example, consider the expression $2 \times 2 \times 3 \times 3 \times 2 \times 3 \times 3$. Writing it as $2^2 \times 3^2 \times 2 \times 3^2$ would be an example of exponent form. However, often we want to write it in an even shorter way. For example that same expression could be written as $2^3 \times 3^4$. In this case we combined all of the twos together and all of the threes together. When we write an expression in exponent form so that each number occurs as a base only once we will say the expression is in **complete exponent form**.

Example 5.1.5. Write the expression $4^3 \times 3^2 \times 5 \times 4^2 \times 3$ in a factored form.

Possible Solution. A factored form of the expression is $4 \times 4 \times 4 \times 3 \times 3 \times 5 \times 4 \times 4 \times 3$. □

Example 5.1.6. Write the expression $4^3 \times 3^2 \times 5 \times 4^2 \times 3$ in a complete exponent form.

Possible Solution.

$$\begin{aligned}
&4^3 \times 3^2 \times 5 \times 4^2 \times 3 \\
&= 4 \times 4 \times 4 \times 3 \times 3 \times 5 \times 4 \times 4 \times 3 \\
&= 4 \times 4 \times 4 \times 4 \times 4 \times 3 \times 3 \times 3 \times 5 \\
&= 4^5 \times 3^3 \times 5
\end{aligned}$$

A complete exponent form for the given expression is $4^5 \times 3^3 \times 5$. □

Note that in the previous example we did *not* use any properties of exponents. Our solution relied solely on the definition of exponents.

5.2 Properties of Exponents

Now that we have developed some facility with the definition of exponents we are ready to begin looking for shortcuts. When students are taught exponents they are often presented with rules to use when computing. Unfortunately if they just try to memorize the rules they will invariably misuse them later on. Therefore we again seek to work through examples that will help us see that exponent rules can be figured out. Teachers do not need to tell students the rules, but instead give examples that allow the students to figure out the important properties of exponents.

Example 5.2.1. Write the expression $5^2 \times 5^6$ in a complete exponent form.

Possible Solution. We will use colors to show how our work is proceeding.

$$\begin{aligned}
&5^2 \times 5^6 \\
&= {\color{blue}5^2} \times {\color{red}5^6} \\
&= {\color{blue}5 \times 5} \times {\color{red}5 \times 5 \times 5 \times 5 \times 5 \times 5} \\
&= 5^8
\end{aligned}$$

Therefore the expression $5^2 \times 5^6$ written in complete exponent form is 5^8. □

Example 5.2.2. Write the expression $3^2 \times 3^4 \times 3^5$ in complete exponent form.

Possible Solution. We will use colors to show how our work is proceeding.

$$\begin{aligned}
&3^2 \times 3^4 \times 3^5 \\
&= {\color{blue}3^2} \times {\color{red}3^4} \times {\color{green}3^5} \\
&= {\color{blue}3 \times 3} \times {\color{red}3 \times 3 \times 3 \times 3} \times {\color{green}3 \times 3 \times 3 \times 3 \times 3} \\
&= 5^{11}
\end{aligned}$$

Therefore the expression $3^2 \times 3^4 \times 3^5$ written in complete exponent form is 3^{11}. □

After doing several examples of this form students will start to recognize the relationship between the exponent in the final answer and the exponents in the original expression. Not only will they be able to see that final exponent came from adding the original exponents, but they will be able to reason out within their work why that is the case. In other words, through well-chosen examples students will discover the additive law of exponents.

Property 5.2.3 (Additive Law of Exponents)**.** For any numbers a^n and a^m, we have the following equation.

$$a^n \times a^m = a^{n+m}$$

Question 5.2.4. How does the work in the examples above justify the Additive Law of Exponents?

The Additive Law of Exponents (abbreviated ALE) is probably the most important exponent rule students will learn, but there are certainly others we would like them to know. Because we have justified ALE, we can now use it moving forward. When working on exponent problems at the beginning of this section we only had one tool in our box. Namely, the definition of exponents. We have now added the ALE too as well. With these two tools in our box, what other properties can we develop?

Question 5.2.5. How can we use the definition of exponents to find the factored form of $\left(3^5\right)^2$?

Once we have answered the question above, we should consider more examples of that form. For example, $\left(2^3\right)^4$ or $\left(7^5\right)^3$. After doing a few of these examples, we should be able to start making generalizations about problems of this form. Now, you may remember the exponent rule that says what $(a^n)^m$ equals. If you don't, complete the examples above and figure out what the rule must say by looking at your work. Either way, once we write down the property below we need to be able to justify that the property holds. This justification shows up in the work we've done with the examples. Again, this is a spot where students do not need to be told the rule. Okay, I wanted to stall enough to give you time to think before reading the rule below. We now state what is sometimes referred to as the Power of a Power rule.

Property 5.2.6. [Power of a Power] For any numbers a, n, and m, (as long as a^n and $a^{n\times m}$ are defined) we have the following equation.

$$(a^n)^m = a^{n\times m}$$

Question 5.2.7. Where in the work on the previous examples would we be able to justify why we are multiplying the exponents? In other words, how we can we use those examples to see why Property 5.2.6 holds?

There may be one more rule that you remember learning about exponents. Just as ALE looked at how exponents behave in certain multiplication problems, we can also look at how exponents behave in division problems. Let's consider the problem $5^7 \div 5^3$. Now we certainly could figure out what 5^7 is (it's 78125) and figure out what 5^3 is (its 125) and then divide them. So $5^7 \div 5^3 = 625$. However, if we are trying to discover a rule about dividing with exponents perhaps staying in exponent form will be of more use.

Question 5.2.8. Suppose we want to stay in exponent form. How could we use the definition of division and the exponent rules we know so far to compute $5^7 \div 5^3$?

Through the work above we find that $5^7 \div 5^3 = 5^4$. More importantly, if we focus on how we got the 4 and perhaps look at a few more examples, then we should be able to come up with the rule for division. We state that rule below.

Property 5.2.9. [Subtractive Law of Exponents] For any numbers a^n and a^m, we have the following equation.

$$a^n \div a^m = a^{n-m}$$

As we proceed through this book, we can only use methods of justification that have been built on prior knowledge. In this book that means if we want to justify something in the integers, we must use our understanding of whole numbers to build on because those are the only other numbers we have dealt with so far. However, it is worth noting that students do in fact learn fractions before exponents, so students are not tied to only using whole numbers. Therefore there are some instances where it is actually easier to justify using fractions rather than whole numbers. One of those instances is in justifying Property 5.2.9. Therefore we give an alternate justification using fractions.

Reader be warned though! This is just an aside. You are not allowed to use fractions from here on out. Notice that we did give a justification above using only whole numbers. This is just an alternate justification that could be useful in the classroom because in that setting students have already studied fractions.

Let's consider the example $5^7 \div 5^3$. The first step is to use the fact division and fractions are interchangeable. Thus $5^7 \div 5^3 = \frac{5^7}{5^3}$. Now let's write this out in factored form. Thus we have $\dfrac{5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5 \times 5}$. In fractions, we can cancel any common factors out on the top and the bottom. Thus the 3 factors of 5 on the bottom will cancel with 3 factors of 5 on top. Thus leaving us with $5 \times 5 \times 5 \times 5$, or 5^4. Note that within this work we can see where the $7 - 4$ appears.

All of the properties discussed about fractions in the above paragraph will be thoroughly addressed in the fractions section. Okay, now the aside is done. We may not use fractions moving forward until the fractions section.

5.3 Exponents Looking Forward

We end this chapter by going back to the definition. You may not have noticed in the statement of the definition of exponents it says that n must be a natural number. So what about exponents like 0 or -2 for example?

Question 5.3.1. If we allowed $^-2$ to be an exponent in the definition of exponents as stated, what would be the problem?

Thus we see that our definition only takes us so far. Recall that this happened when we were multiplying two negative numbers. The definition of multiplication no longer applied in that setting so we had to use properties that we knew about multiplication to determine what a negative times a negative was. The same idea is true here. The definition of exponents does not apply to an exponent of $^{-}2$, so we will have to use properties that we know about exponents to determine what 5^{-2} equals for example. We will address negative exponents in a later section.

Let's instead focus on an exponent of 0. Imagine if a student were trying to use the definition to make sense of 5^0 for example. What do you think students might say 5^0 is equal to?

Since we cannot apply the definition of exponents to 5^0, then just like we did when multiplying two negatives, we need to be clever and use prior properties to reason out what 5^0 must equal.

Question 5.3.2. How can we figure out what 5^0 must equal?

5.4 Exercises

1. (a) Explain to Max what the Additive Law of Exponents is. (Do not explain why it works.)
   (b) Explain to Max why the Additive Law of Exponents works.
   (c) Max knows that when he wants to compute $\left(2^3\right)^4$ he is supposed to use the rule that says to multiply the exponents. However, he doesn't understand why the exponents are being multiplied. Explain to Max how to use the Additive Law of Exponents to see why the exponents are being multiplied.

2. (a) Explain why we cannot use the definition of exponents to understand what 2^0 equals.
   (b) Use the Additive Law of Exponents to determine what $2^0 \cdot 2^3$ equals.
   (c) Use your result from part 2b to justify why 2^0 must equal 1.
   (d) Use the Additive Law of Exponents to determine what $2^3 \cdot 2^{-3}$ equals.
   (e) Use your result from part 2d to determine what 2^{-3} must equal.

3. (a) What is the difference between -3^2 and $(-3)^2$?
   (b) Suppose a is some non-zero integer. Consider the statement: a^2 is positive. Is this statement always true, sometimes true, or never true? Justify your response.

4. Suppose a and b are integers. Are the following statements always true, sometimes true, or never true? Justify your response.

   (a) $(a+b)^2 = a^2 + b^2$
   (b) $(ab)^2 = a^2b^2$

5. In algebra, students learn how to simplify expressions like $3x \cdot 5x^2$ for example. Give a careful, step by step justification that includes the names of properties you used to show that $3x \cdot 5x^2$ is equal to $15x^3$.

6. Suppose a and b are integers such that $a < b$. Determine whether the following statements are always true, sometimes true, or never true.

   (a) $a^2 < b^2$
   (b) $a^3 < b^3$

6 Number Theory

6.1 Fundamental Theorem of Arithmetic

I have heard more than one argument about what a prime number is. The argument always revolves around the number 1. Is 1 prime or not? Well, as usual, let's begin with a definition.

Definition 6.1.1. A natural number a is considered to be a prime number if it has exactly two positive factors.

Before we discuss the issue of one we should probably review the word factor. The definition of a factor was covered in a previous course but we recall it here to avoid any confusion.

Definition 6.1.2. We say a is a factor of b if there is an integer n so that $b = a \times n$.

For example, 4 is a factor of 12 because 3 is an integer and $12 = 4 \times 3$. Notice that if $12 = 4 \times 3$, then $12 \div 4 = 3$. What that means then is that we may also think of a factor from a division standpoint. So an alternate interpretation that is equivalent to the definition above is as follows. We can say that a is a factor of b if $b \div a$ is an integer. This alternate interpretation is used frequently. For example, we may hear a student say that 4 is a factor of 12 because 4 goes into 12 evenly. The situation in which we are working will dictate whether we use the multiplication interpretation or the division interpretation of factor.

Because they go hand in hand we will recall the definition of a multiple as well. The definition uses the same equation, but focuses on a different part of the equation. Let's first look at the example of 4 and 12. Above we said that 4 is a factor of 12 because $12 = 4 \times 3$. If instead we focus on the 12, we call that a multiple of 4. The definition is as follows.

Definition 6.1.3. We say that c is a multiple of d if there is an integer m so that $c = d \times m$.

Example 6.1.4. List all positive factors of 36.

Answer. The positive factors of 36 are $1, 2, 3, 4, 6, 9, 12, 18,$ and 36.

Example 6.1.5. What is the smallest three digit number that is a multiple of 36?

Possible Solution. We know that we will have a multiple of 36 if that number is equal to 36 times some integer. For example $36 \times 2 = 72$. Thus 72 is a multiple of 36, but 72 is not a 3 digit number. How about 36×3? That equals 108! Thus 108 is the smallest three digit number that is a multiple of 36. □

Question 6.1.6. Why would it be unfair to ask you to list all positive multiples of 36?

Example 6.1.7. Suppose you have just figured out that $4424 = 56 \times 79$. Without using a calculator or writing anything down, determine whether or not 7 is a factor of 4424.

Possible Solution. The problem says "without writing anything down", but of course I have to write something down in order to communicate what could be done in my head to answer this question.

We know $4424 = 56 \times 79$. However, we also know that $56 = 7 \times 8$, so $4424 = 7 \times 8 \times 79$. Now, we don't know what 8×79 is, but we do at least know it is an integer. Therefore we were able to write 4424 as 7 times an integer, and so 7 is a factor of 4424. A more visual description of this thought process is shown below.

$$4424 = 56 \times 79$$

$$\downarrow$$

$$4424 = \overbrace{7 \times \underbrace{8}\!\!\!\!\quad\!\! \times 79}$$

$$4424 = \overbrace{7 \times 8} \times 79, \qquad 4424 = 7 \times \underbrace{8 \times 79}$$

$$\downarrow$$

$$4424 = 7 \times \text{some integer}$$

Since we were able to write 4424 as 7 times some integer, then we know that 7 is a factor of 4424. □

Notice that just with the work we've done above we can say several more things. First of all we also know 8 is a factor of 4424 because $4424 = 8 \times 7 \times 79$ and in this case the "some integer" is 7×79. We also know that 56 and 79 are also factors of 4424. Looking at the multiples instead of factors we know that 4424 is a multiple of 56, of 79, of 7, and of 8.

Because multiples and factors are so closely related, students often get them mixed up. It is always important that we are precise with our language, but in settings where we know students tend to get confused we want to be extra careful with our language. That brings us back to the argument about 1 being prime or not. Look back at the very precise language in the definition.

Question 6.1.8. According to the definition of a prime number, is 1 prime?

Prime numbers are the building blocks for natural numbers. Wait a minute...precise language! What do we mean by that? Well, let's consider a number that is not prime, say 12. Since it is not prime, it has factors other than 1 and 12. That means we can break 12 down into a product of two numbers that are both smaller than 12. For example, $12 = 3 \times 4$. We know that 3 is prime, but 4 is not so we can break 4 down into the product of two numbers that are smaller than 4. In particular, $4 = 2 \times 2$. Thus $12 = 3 \times 2 \times 2$. We can not break down any further because all of numbers are prime. Therefore when we say prime numbers are the building blocks, more precisely what we mean is that any natural number greater than 1 that is not prime can be broken down into a product of prime numbers. This is a super important theorem in mathematics, and thus is given a special name.

Theorem 6.1.9. ***The Fundamental Theorem of Arithmetic***

Every natural number greater than 1 is either prime or is equal to a product of prime numbers. In addition, the product of primes is unique up to reordering.

The first sentence is exactly what we were talking about in the previous paragraph. Twelve is not prime so it can be written as a product of primes, namely $12 = 3 \times 2 \times 2$. The second sentence in the theorem is subtle but extremely important. Here's what it is saying. Suppose you and I decide to write the number 18 as a product of primes. We most likely will not have exactly the same work and in fact our final answer may look different. But what that second sentence is saying is that it won't look much different. The only difference in our answer should be the order we chose to write them in. For example, I may write $18 = 3 \times 2 \times 2$ and you may write $18 = 2 \times 2 \times 3$. We wrote them in a different order but the numbers we came up with were exactly the same. And the amount of each is exactly the same. In other words, whoever decides to write 18 as a product of primes their answer will contain exactly two 2s and one 3.

In this section, we will use the fact that we can break numbers down in this way to discover some interesting properties about natural numbers. In our work we will want to distinguish between numbers that are prime and those that aren't. When a natural number greater than 1 is not prime, we say that number is **composite**. For example, 12 is a composite number. So composite just means the number has factors other than 1 and itself.

When we write a composite number as a product of primes, we call that the **prime factorization** of the number. If there are repeated factors, we often write it in exponent form. For example, the prime factorization of the number 12 is $2 \times 2 \times 3$. If instead we write it in exponent form, then the prime factorization of 12 is $2^2 \times 3$.

6.2 Finding Factors Using Prime Factorization

Let's explore how the prime factorization of a number can help us find factors of that number.

Example 6.2.1. Find the prime factorization of the number 1176.

Possible Solution. We will use our calculator when needed to help us factor 1176. Since the number is even we know it is divisible by 2. Computing $1176 \div 2$ we get 588. This also is even so we divide 588 by 2, giving us 294. Even again, so dividing 294 by 2 we get 147. With all that work, so far we know that $1176 = 2 \times 2 \times 2 \times 147$. Let's see if we can factor 147. Try 3...yes, $147 \div 3 = 49$. So now we know $1176 = 2 \times 2 \times 2 \times 3 \times 49$, and of course we can factor 49, so we end up with the following factorization (written in both factored and complete exponent form).

$$1176 = 2 \times 2 \times 2 \times 3 \times 7 \times 7 = 2^3 \times 3 \times 7^2$$

□

Now that we have factored 1176 into primes, let's see how we can use that to our advantage when trying to find factors. It may be helpful to look back at Example 6.1.7 because we will use similar logic in the following example.

Example 6.2.2. Without using a calculator or writing anything down determine if 21 is a factor of 1176.

Possible Solution. In order to determine if 21 is a factor of 1176, we would need to figure out if 1176 is equal to 21 times some integer. Well, let's see what we can do with the prime factorization from above. (Note that the work shown below can all be done in our head, so we are not breaking the rule of not writing anything down.)

$$\begin{aligned}1176 &= 2 \times 2 \times 2 \times 3 \times 7 \times 7 \\ &= 3 \times 7 \times 2 \times 2 \times 2 \times 7 \\ &= 21 \times \underbrace{2 \times 2 \times 2 \times 7} \\ &= 21 \times \text{some integer}\end{aligned}$$

Therefore 21 is a factor of 1176. □

Question 6.2.3. Is 56 a factor of 1176?

What about 63? Is 63 a factor of 1176? Believe it or not, this is where that second sentence in the Fundamental Theorem of Arithmetic is important. Let's first think about 63. It's prime factorization is $3^2 \times 7$. In other words, 63 is made up of two 3s and a 7. If 63 is a factor of 1176, then we would be able to write 1176 as 63 times some integer. However, if we could do that then when we broke 63 and the other integer down into primes it would give us a prime factorization of 1176. But that would mean that the prime factorization of 1176 would have at least two 3s (from the 63). However, we already have a prime factorization that has only one 3. The Fundamental Theorem of Arithmetic says we can't have two different prime factorizations, so 1176 can't have a prime factorization with two 3s and so 63 can't be a factor of 1176.

Whew! That was some pretty deep mathematical thinking there. But the cool thing that deep thinking gave us is that we can completely determine factors of a number by looking at its prime factorization. Not only can we tell if a number is a factor we can also tell if a number is not a factor. That's some pretty powerful mathematics right there!

We will come back to the power of prime factorization in a moment, but for the following example the numbers are small enough where we don't really need to look at their prime factorizations to answer the questions. Of course, we certainly can if we want to though.

Example 6.2.4. How many positive factors does 25 have?

Answer. The positive factors of 25 are 1, 5, and 25, so 25 has 3 factors.

Example 6.2.5. How many positive factors does 24 have?

Possible Solution. The positive factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24, so 24 has 8 factors.

Looking back at the two examples, you may be surprised by the result. The numbers 24 and 25 are obviously very close to each other, but 24 had way more factors than 25. Why do you suppose that is? Well, that is definitely a question worthy of exploration. Perhaps it is time to go back to prime factorizations...

Example 6.2.6. Consider the number N whose prime factorization is $2^5 \times 3^6 \times 7^2 \times 13$. Which of the following numbers are factors of N? (Do not use a calculator.)

$2^4 \times 3$	72	2×7^3	3^7
13^3	$2 \times 7 \times 13$	$2^5 \times 3^6 \times 7^2 \times 13$	35

Answer. Four of the numbers above are factors of N. They are: $2^4 \times 3$, 72, $2 \times 7 \times 13$, and $2^5 \times 3^6 \times 7^2 \times 13$. □

Question 6.2.7. Consider the number $A = 2^3 \times 7^2$. Do not actually compute the value of A. Write down (in exponent form) all the positive factors of A.

Let's ask the same question but with a different number. In completing the next task it may be helpful for us to pay attention to the similarities and differences between the two questions and solutions.

Question 6.2.8. Consider the number $B = 3^2 \times 5^3$. Do not actually compute the value of B. Write down (in exponent form) all the positive factors of B.

After answering the two questions above, we should have found that A and B have the same number factors. Notice that the factors themselves were not the same, but the amount of factors was.

Question 6.2.9. Looking at A and B above, can you come up with your own number, C, that would have the same number of positive factors as A and B?

Example 6.2.10. Consider the number $D = 5 \times 11^4$. Do not actually compute the value of D. How many positive factors does D have?

Answer. (You should have your own organized way of listing out the factors that makes sense to you.)

D has 10 positive factors. □

At this point we should have our own way of organizing and listing out factors of a given number. However, notice the numbers A, B, and D above. They all have something in common...each of their prime factorizations contain only two distinct primes. Hmm...what happens if there are more?

Example 6.2.11. Consider the number $E = 3^2 \times 5 \times 11^4$. Do not actually compute the value of E. How many positive factors does E have?

Answer. (You should have your own organized way of listing out the factors that makes sense to you.)

E has 30 positive factors. □

After doing these examples (and perhaps a few more) we should be able to generalize our findings. In other words, we should be able to describe a method that allows us to determine the number of factors there will be using the prime factorization.

Question 6.2.12. Suppose we have found a factor of some number T. What can we say about the prime factorization of the factor in relation to the prime factorization of T?

Question 6.2.13. How can we use the prime factorization of a number to determine how many factors that number will have?

6.3 Finding Multiples Using Prime Factorization

In the previous section, we spent a great deal of time exploring how useful the prime factorization can be in finding factors of a number. Recall that asking "is a a multiple of b" is *exactly* the same as asking "is b a factor of a". Since factors and multiples are tied we should expect that the prime factorization will help us find multiples as well.

Example 6.3.1. Which of the following numbers are multiples of 24?

2   8   24   56   96

Possible Solution. Recall that a multiple of 24 must be equal to 24 times some integer. We write that equation for each number above and determine if such an integer exists or not.

$2 = 24\times?$	There is definitely no integer that we could replace the question mark with to make this equation true. Therefore 2 is not a multiple of 24.
$8 = 24\times?$	There is definitely no integer that we could replace the question mark with to make this equation true. Therefore 8 is not a multiple of 24.
$24 = 24\times?$	There is an integer that we could replace the question mark with, namely $? = 1$, to make this equation true. In other words, $24 = 24 \times 1$, so 24 is a multiple of 24.
$56 = 24\times?$	Well, $24\times 2 = 48$, which is smaller than 56. And $24\times 3 = 72$, which is larger than 56. Therefore there is no integer that we could replace the question mark with to make this equation true. Therefore 56 is not a multiple of 24.
$96 = 24\times?$	Continuing from our work above we find that $24 \times 4 = 96$. Therefore 96 is a multiple of 24.

Thus, of the numbers listed above, only 24 and 96 are multiples of 24. □

Before going to another example we should summarize and generalize how we proceeded above. When we wanted to determine whether or not a number was a multiple of 24, we just kept asking ourselves: "Is the number equal to 24 times some integer?". Therefore, when looking for multiples of a given number we are looking for that given number times some integer. We apply that technique below, but will be working with the numbers in prime factorization form.

Example 6.3.2. Consider the number $A = 3^5 \times 7^3 \times 11 \times 19^4$. Is the number $3^6 \times 7^5 \times 11 \times 19^8$ a multiple of A?

Possible Solution. We should be asking ourselves whether or not $3^6 \times 7^5 \times 11 \times 19^8$ is equal to A times some integer. In other words, can we pull an A out of this number?

$$
\begin{aligned}
& 3^6 \times 7^5 \times 11 \times 19^8 \\
&= \overbrace{3^5 \times 3} \times \overbrace{7^3 \times 7^2} \times 11 \times \overbrace{19^4 \times 19^4} \\
&= \underbrace{3^5 \times 7^3 \times 11 \times 19^4} \times \underbrace{3 \times 7^2 \times 19^4} \\
&= \qquad A \qquad \times \text{ some integer}
\end{aligned}
$$

Therefore we were able to write the number $3^6 \times 7^5 \times 11 \times 19^8$ as A times some integer, so $3^6 \times 7^5 \times 11 \times 19^8$ is a multiple of A. □

Example 6.3.3. Consider the number $S = 3^5 \times 5^4 \times 7$. Do not find the value of S. Which of the following numbers are multiples of S?

$a = 3^6 \times 5^4 \times 7^3$ $\quad b = 2^2 \times 3^6 \times 5^4 \times 7^3$ $\quad c = 3^4 \times 7^8$

$d = 3^6 \times 7^3$ $\quad e = 3^8 \times 5^2 \times 7^2 \times 11^3 \times 19$ $\quad f = 2^4 \times 3^5 \times 5^4 \times 7 \times 13$

Answer. Of the numbers given above, only a, b, and f are multiples of S. □

Question 6.3.4. Suppose we have found a multiple of some number T. What can we say about the prime factorization of the multiple in relation to the prime factorization of T?

6.4 GCF and LCM Using Prime Factorization

In the previous section we were looking at factors and multiples of a single number. We move now to explore **common factors and multiples**. In other words, factors and multiples that two or more numbers have in common. In particular, we will focus on finding common factors and multiples using prime factorizations. Before doing so, however, let's be sure we are comfortable with the notion of common factors and multiples.

Example 6.4.1. Is 12 a common factor of 36 and 40?

Answer. No. Twelve is a factor of 36 but not of 40.

Example 6.4.2. Is 4 a common factor of 36 and 40?

Answer. Yes. Four is a factor of both 36 and 40.

Example 6.4.3. Is 70 a common multiple of 36 and 40?

Answer. No. Seventy is not a multiple of 40 nor of 36.

Example 6.4.4. Is 720 a common multiple of 36 and 40?

Answer. Yes. 720 is a multiple of both 36 and 40.

Example 6.4.5. Is 4 a common factor of 36, 40, and 45?

Answer. No. Four is a factor of 36 and 40 but not of 45.

Example 6.4.6. What is the largest number that is a common factor of 36 and 40?

Possible Solution. The positive factors of 36 are $1, 2, 3, 4, 6, 9, 12, 18$, and 36. The positive factors of 40 are $1, 2, 4, 5, 8, 10, 20$, and 40. The only positive factors that 36 and 40 have in common are $1, 2$, and 4. Therefore 4 is the largest number that is a common factor of 36 and 40. □

Example 6.4.7. What is the smallest positive number that is a common multiple of 36 and 40?

Possible Solution. A list of the first several positive multiples of 36 and 40 are shown below.

$$\text{Multiples of 36: } 36, 72, 108, 144, 180, 216, 252, 288, 324, 360, 396, \ldots$$

$$\text{Multiples of 40: } 40, 80, 120, 160, 200, 240, 280, 320, 360, 400, 440, \ldots$$

As we work our way up each list we see that 360 is the first number that is a multiple of both 36 and 40. Therefore 360 is the smallest positive number that is a common multiple of 36 and 40. □

Although we do assume prior knowledge of factors and multiples we want to again be sure we are on solid footing, so let's take a few moments to discuss some of the examples above. Notice that in Example 6.4.6 we asked for the largest number that is a common factor of the given numbers. Recall that we call this the **greatest common factor** (GCF). It is worth pondering why the question is always about the *greatest* common factor. Why don't we ask about the *least* common factor?

Similarly, in Example 6.4.7 we asked for the smallest positive number that is a common multiple of the given numbers. Recall that we call this the **least common multiple** (LCM). Again, it is worth pondering why the question is always about the *least* common multiple. Why don't we ask about the *greatest* common multiple? Also, why did we need to include the word *positive* in that question?

After solidifying our understanding of GCF and LCM by answering the questions above, we are now ready to notice some difficulties with the way we computed our answers in the previous examples. In particular, we listed out all the positive factors of each number first and then looked for commonalities. Similarly, for multiples we listed a bunch of multiples of each number until we came across a commonality. This could get very long and tedious. But not to worry, prime factorization is here to save the day! And the best part is we have already done all of the hard work in the previous sections, so all we need to do is apply it strategically.

For each of the examples below we will use the numbers A and B whose prime factorizations are given below.

$$A = 2^3 \times 3^5 \times 7 \qquad B = 2^5 \times 3 \times 5^2$$

Example 6.4.8. Find a value for ? so that $2^?$ is a factor of B but not of A. If it is not possible, explain why not.

Possible Solution. Both 2^4 and 2^5 would be factors of B, but would not be factors of A. Therefore the question mark could be 4 or 5. □

Example 6.4.9. Find a value for ? so that $2^?$ is a factor of A but not of B. If it is not possible, explain why not.

Possible Solution. A factor of A must be made up of the prime numbers occurring in the prime factorization of A. Therefore a factor of A cannot have more than three 2s in its prime factorization. So to be a factor of A the question mark must be 0, 1, 2, or 3. However, if we use any of those values for the question mark, then $2^?$ would also be a factor of B. Therefore we cannot find a value for the question mark so that $2^?$ is a factor of A but not B. □

Example 6.4.10. Find a value for ? so that $2^?$ is a not a factor of A nor of B. If it is not possible, explain why not.

Possible Solution. If we choose a value for the question mark of 6 or larger, then $2^?$ will be a factor of neither A nor B. □

Example 6.4.11. Find a value for ? so that $2^?$ is a factor of both A and B. If it is not possible, explain why not.

Possible Solution. From above we know that to be a factor of A the question mark must be 0, 1, 2, or 3, and those choices all make $2^?$ me a factor of B as well. Thus the question mark could be 0, 1, 2, or 3. □

Example 6.4.12. We want the number $2^? \times 3$ to be a factor of both A and B. Is that possible? If so, what values of ? will make that happen? If not, why not?

Possible Solution. From above we saw that a factor of both A and B cannot have more than three 2s in its prime factorization. In this case though, we also have the 3 to worry about. Since both A and B have at least one 3 in their prime factorization, then the 3 will not cause any problems in the creation of a factor. Therefore the question mark could be 0, 1, 2, or 3. □

Question 6.4.13. We want the number $2^? \times 3^2$ to be a factor of both A and B. Is that possible? If so, what values of ? will make that happen? If not, why not?

After working through these examples (and perhaps a few more) we should be able to generalize a method we could employ to find the GCF of any two numbers by using their prime factorizations. Let's first recap what we learned about factors and prime factorization in Section 6.2. We know that a factor of a number must be made up of prime numbers that appear in the given number. For example, a factor of A can only be made up of prime numbers that appear in the prime factorization of A. So in our example, a factor of A must be made up of only 2s, 3s, and 7s. Moreover, a factor of A must be made up of what is in A so that means a factor of A cannot have more primes in its factorization than A does. In our example then, a factor of A cannot have more than three 2s, five 3s, and one 7.

We can use this idea to build common factors, and in particular to find the largest number that is a common factor (i.e. the GCF). Let's stick with our example for a bit.

Question 6.4.14. What prime numbers can we use in order to build a number that is a factor of both A and B?

Question 6.4.15. How do we use those prime numbers to build the largest number that is a factor of both A and B?

A careful analysis of the previous two questions should give rise to a general method for finding GCF using the prime factorization. A good test of whether or not we truly understand an idea or concept is to see if we can put it into our own words rather than just stating some rule.

Question 6.4.16. What is your explanation of how to find the GCF of any two numbers using the prime factorization?

Notice that we did not have to develop any new understanding or techniques to find the GCF. Rather we just relied on our understanding of how to find factors of a single number and then applied it to a pair of numbers. We will proceed similarly in finding the LCM using prime factorization.

For each of the examples below we will again use the numbers A and B whose prime factorizations are given below.

$$A = 2^3 \times 3^5 \times 7 \qquad\qquad B = 2^5 \times 3 \times 5^2$$

Example 6.4.17. Find a value for ? so that $2^? \times 3^6 \times 5^4 \times 7$ is a multiple of A but not of B. If it is not possible, explain why not.

Possible Solution. To be a multiple of A we must have everything in A's prime factorization show up in the multiple. In other words, to be a multiple of A we must have at least three 2s, five 3s, and one 7. The number $2^? \times 3^6 \times 5^4 \times 7$ has six 3s and one 7, so we have enough 3s and 7s. Therefore we just need to choose the question mark to have enough 2s. Since we want at least three 2s, then we can choose our question mark to be 3 or bigger so that we have a multiple of A.

Using the same argument as above, to be a multiple of B we must have at least five 2s, one 3, and two 5s. The number $2^? \times 3^6 \times 5^4 \times 7$ has six 3s and four 5s, so we have enough 3s an 5s. Therefore the amount of 2s is the only thing that can keep $2^? \times 3^6 \times 5^4 \times 7$ from being a multiple of B. Since a multiple of B needs at least five 2s, we want to choose a value for the question mark that is less than 5, to insure we do not have a multiple of B.

Therefore to be a multiple of A we need the question mark to be 3 or greater, but to insure it is not a multiple of B we need to make sure the question mark is less than 5. Thus the value of the question mark could be 3 or 4. ☐

Example 6.4.18. Find a value for ? so that $2^? \times 3^6 \times 5^4 \times 7$ is a multiple of B but not of A. If it is not possible, explain why not.

Possible Solution. Using the same argument as above, we know to be a multiple of B we need the question mark to be 5 or greater. However, if the question mark is 5 or greater, then $2^? \times 3^6 \times 5^4 \times 7$ will be a multiple of A as well. Therefore we cannot choose a value for the question mark that would make the number a multiple of B and not A. ☐

Example 6.4.19. Find a value for ? so that $2^? \times 3^6 \times 5^4 \times 7$ is a not a multiple of A nor of B. If it is not possible, explain why not.

Possible Solution. Again, from above we know to be a multiple of A we need the question mark to be 3 or greater and to be a multiple of B we need the question mark to be 5 or greater. Therefore to insure that it is a multiple of neither A nor B we could choose the question mark to have a value of 0, 1, or 2. □

Example 6.4.20. Find a value for ? so that $2^? \times 3^6 \times 5^4 \times 7$ is a multiple of both A and B. If it is not possible, explain why not.

Possible Solution. From our discussion above, the lowest value for the question mark to insure a multiple of B is 5. And the value of 5 or higher for the question mark would also make $2^? \times 3^6 \times 5^4 \times 7$ be a multiple of A as well. Therefore if the question mark is 5 or greater we will have a multiple of both A and B. □

Example 6.4.21. We want the number $2^? \times 3^4 \times 5^4 \times 7$ to be a multiple of both A and B. Is that possible? If so, what values of ? will make that happen? If not, why not?

Possible Solution. Notice that we changed things up just a bit. The new number we are looking at as a potential multiple of both A and B only has four 3s, whereas in the previous examples, the number in question had six 3s. We need to think about how the change in the number of 3s affects things. To be a multiple of A we would need at least five 3s, which we don't have. In other words, we don't have enough 3s to be a multiple of A. Therefore no matter what value we pick for the question mark we will never have enough 3s. Therefore there is no value for the question mark that would make $2^? \times 3^4 \times 5^4 \times 7$ a multiple of A. □

Question 6.4.22. We want the number $2^? \times 3^5 \times 7$ to be a multiple of both A and B. Is that possible? If so, what values of ? will make that happen? If not, why not?

After working through these examples (and perhaps a few more) we should be able to generalize a method we could employ to find the LCM of any two numbers by using their prime factorizations. Let's first recap what we learned about multiples and prime factorization in Section 6.3. Any multiple of a given number must have a prime factorization that contains, at the very least, the prime factorization of the given number. For example, a multiple of A must have a prime factorization that contains at least three 2s, five 3s, and one 7. Note that this is the very least the multiple can have, but it certainly can have more than that. In particular, it can even have other prime numbers. For example, the number whose prime factorization has five 2s, five 3s, two 7s, three 5s, and two 13s would also be a multiple of A. The key here is that a multiple of A has minimum requirements on 2s, 3s, and 7s in its prime factorization.

We can use this idea to build common multiples, and in particular to find the smallest number that is a common multiple (i.e. the LCM). Let's stick with our example for a bit.

Question 6.4.23. What prime numbers can we use in order to build a number that is a multiple of both A and B?

Question 6.4.24. What prime numbers must we use in order to build a number that is a multiple of both A and B?

Question 6.4.25. How do we build the smallest number that is a multiple of both A and B?

A careful analysis of the previous three questions should give rise to a general method for finding the LCM using the prime factorization. As mentioned above, a good test of whether or not we truly understand an idea or concept is to see if we can put it into our own words rather than just stating some rule.

Question 6.4.26. What is your explanation of how to find the LCM of any two numbers using the prime factorization?

6.5 Exercises

1. Consider the number 539. Note that $539 = 7^2 \times 11$.

   (a) How many different values of a can you choose so that 7^a is a factor of 539? Explain how you arrived at your answer.

   (b) How many different values of b can you choose so that 11^b is a factor of 539? Explain how you arrived at your answer.

   (c) As discussed in class, we know that any factor of 539 must be made up of sevens and elevens. In other words, any factor of 539 can be expressed as $7^x \times 11^y$ for some x and y. Use this fact and your work in the previous two parts to determine how many factors 539 will have without actually finding any of them. Explain how you arrived at your answer.

2. Max says that the bigger the number, the more factors it has. Is Max correct? If yes, why? If not, provide a counterexample to show he is not correct.

3. Consider the numbers A and B below.

$$A = 2^{35} \times 5^{12} \times 7^{21} \qquad B = 2^{23} \times 3^7 \times 7^{32}$$

   (a) Consider the number 2^n. If possible, find a value of n so that 2^n is a factor of A but not of B. If it is not possible, why not?

   (b) Consider the number 2^n. If possible, find a value of n so that 2^n is a factor of B but not of A. If it is not possible, why not?

   (c) Consider the number 2^n. What is the largest value of n you could choose so that 2^n is a factor of both A and B.

   (d) Explain how to use the prime factorization of A and B to find the greatest common factor of A and B.

   (e) After hearing your explanation, Max asks, "Why do we take the **smaller** exponent when we are finding the **greatest** common factor? Shouldn't we take the **larger** exponent?" Explain to Max why we take the smaller exponent.

4. Consider the numbers A and B below.

$$A = 2^{35} \times 5^{12} \times 7^{21} \qquad B = 2^{23} \times 3^7 \times 7^{32}$$

   (a) Find a number that is a multiple of A but not of B. (You may just provide the prime factorization of the number.)

(b) Find a number that is a multiple of B but not of A. (You may just provide the prime factorization of the number.)

(c) Find a number that is neither a multiple of A nor of B. (You may just provide the prime factorization of the number.)

(d) Find a number that is a multiple of both A and B, but is not the least common multiple. (You may just provide the prime factorization of the number.)

(e) Explain how to use the prime factorization of A and B to find the least common multiple A and B.

5. Suppose a is some natural number and that $GCF(a, 12) = 1$. Determine whether the following statements are always true, sometimes true, or never true. Justify your response.

(a) 2 is a factor of a. (b) 5 is a factor of a. (c) 10 is a factor of a.

6. Suppose b is some natural number and that $GCF(b, 12) = 6$. Determine whether the following statements are always true, sometimes true, or never true. Justify your response.

(a) 2 is a factor of b. (b) 4 is a factor of b. (c) 9 is a factor of b.

7 Geometry (Part I)

7.1 Perimeter and Area

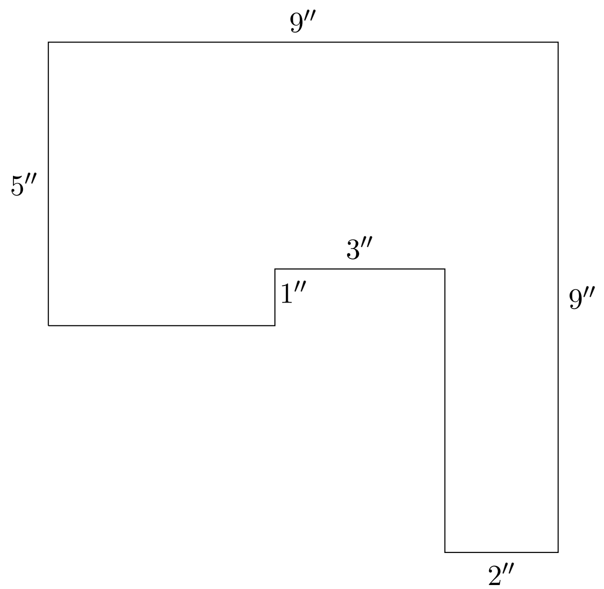
Perimeter and area are topics that are quite relevant to "everyday" life, so most people have a notion of what perimeter and area mean. In order to move forward mathematically in this section however, we need more than a notion of perimeter and area. For example, suppose you are looking at houses for sale on the internet and the description says the house is 1800 square feet. What does that actually mean?

Both perimeter and area are measurements of some sort. Perimeter, which is the distance around a shape, is a measurement of length then. Area on the other hand measures how much can fit inside the shape. In order to really be precise about area, we are first going to be really precise (and you'll probably think somewhat silly) about perimeter. Suppose we said that the perimeter of some rectangle is 18 feet. Let's imagine a martian trying to decipher what that means. Well, our friendly martian would first have to understand that perimeter just means the distance you would travel if you walked along all the edges of the shape. But the martian would also need to understand what 18 feet means. In order for her to understand this we would have to show her what 1 foot looks like by showing her a stick whose length is 1 foot. She then can imagine needing 18 of those pieces to fit around the shape. Granted we may have to break some of those pieces at corners, but in total we would end up using all 18 pieces and they would end up covering all the edges.

Notice that what the martian needed to understand is what a single piece looked like and how many of those pieces we need to cover the part of the shape we were interested in. For perimeter, we were covering the edges and a single piece was just a line segment (i.e. a stick). Therefore to really understand area we need to know what we are covering and with what. We are going to assume we are working in just two dimensions for now. So any shape we are discussing can be drawn on a piece of paper. For area then, we want to cover the region inside the shape. Cover them with what though? Well, let's go back to the example of the house. Notice the units were square feet. So there's our answer. Rather than a stick of length one, we want a square whose sides are length one. In other words, a square foot is a 1 foot by 1 foot square. We have completely described area now. The area of a shape is the number of 1 unit by 1 unit squares we can fit inside the shape. So in the example of the house if we made 1800 squares that were all 1 foot by 1 foot, then we would be able to completely cover the floor of the house. Again, we may have to break some of those squares, but we will use them all up and cover the whole floor.

Now that we have a solid understanding of what perimeter and area actually are, let's take a look at some examples. Our eventual goal will be to find some shortcuts for finding areas (in other words, area formulas), so as of right now we may only use the definitions of perimeter and area described above. To be clear, you have no formulas at your disposal at this moment. (Don't worry that will change quite soon.)

Example 7.1.1. Find the perimeter of the shape described below. (You may assume every corner is a right angle.)



Answer. The perimeter is 38 inches.

Example 7.1.2. Find the area of the shape described above.

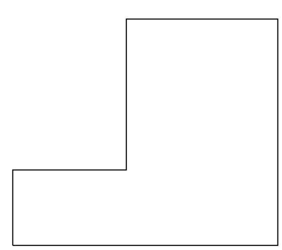
Possible Solution. After attempting the problem on your own, take a look at the video solution below.

Video Solution to Area Problem

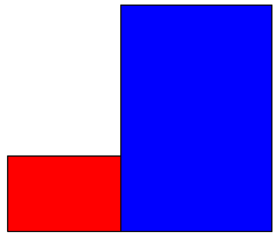
The video shows only one way to compute the area, but of course there are lots of ways to find the area. (Just be sure you are not using any formulas.) All methods should arrive at the answer of 50 square inches for the area. □

The video used an important idea in area that should be called out. To find the area of a shape, we can break it up however we like, find the area each of the pieces, and then add those areas together to get the total area of the shape. When we get to deriving area formulas one of the strategies we will use is this additive areas idea. In other words, we can view a shape as smaller pieces put together. However, there is nothing to stop us from also viewing a shape as a larger shape with smaller pieces removed.

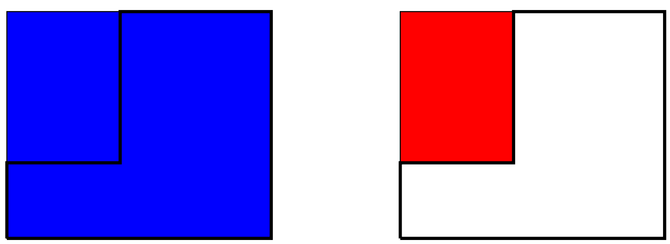
Let's consider the shape below, and think about how a student could view this additively or subtractive.



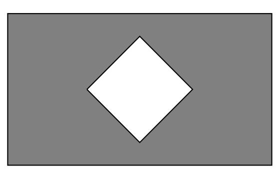
We can view the above shape as two rectangles put together shown in red and blue below. Therefore to find the total area of the shape a student could find the red and blue areas separately, and then add those areas together to get the total area.



On the other hand a student could view the shape as a larger rectangle with a smaller rectangle removed. The larger rectangle is shown in blue below and the rectangle being removed is shown in red. The red and blue areas could be computed separately, and then to find the total area the student could subtract the red area from the blue area.



Example 7.1.3. The rectangle shown below is 4 inches by 7 inches and inside the rectangle is a square whose side length is 2 inches. Find the area of the shaded region.



Possible Solution. The dimensions of the outer rectangle are 4 inches by 7 inches, so that means we can break the rectangle into four rows of seven 1 inch by 1 inch squares, giving us a total of 28 1 inch by 1 inch squares. We are also told the white shape is a square that is 2 inches by 2 inches, and even though it is tilted there will still be 4 1 inch by 1 inch squares to fill the white area. Therefore to get the shaded region we have 28 square inches from the rectangle and we want to remove the 4 square inches that make up the white square. Thus the area of the shaded region is 24 square inches. □

Question 7.1.4. A circle has a radius of 3 feet. Is its area larger, smaller, or equal to 36 square feet? (Remember you do not have any area formulas to work with.)

7.2 Formulas for Area

We now want to work on developing formulas to compute the areas of some standard shapes. We will start with a rectangle. Of all the area formulas, the rectangle formula is the one most commonly remembered. Namely that the area of a rectangle is the length times the width. In formula notation, we have $A = lw$.

Question 7.2.1. Why does computing length times width find the area of a rectangle?

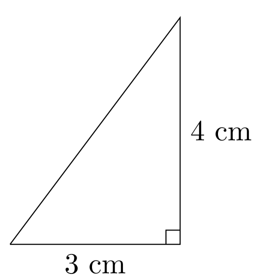
We will assume we now have the formula for the area of a rectangle and will use it at will moving forward. Having this formula under our belt, let's move on to parallelograms. Just as a reminder, a parallelogram is a quadrilateral where each pair of opposite sides are parallel. (A rectangle is a special type of parallelogram.)

Question 7.2.2. How could we use the area formula for rectangles to develop a formula for the area of a parallelogram?

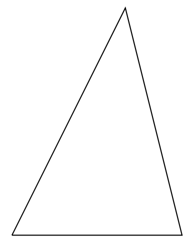
When the formula for a parallelogram is stated in textbooks it is usually stated as base times height. In other words, $A = bh$. This language is different that the length and width we used for rectangles. In rectangles, it is clear what we mean by length and width. (Although people sometimes argue about which one is length and which is one is width. Of course that's just silly, because thanks to the commutative property of multiplication, who cares!) However, this new language of *base* and *height* do deserve some discussion. However, this language gets used in the formula for the area of a triangle as well, so we will postpone that discussion until after looking at triangles.

The formula for the area of a triangle is probably the second most common formula that people remember. Namely, that the area of a triangle can be found by computing one-half multiplied by the base multiplied by the height. In formula notation, we have $A = \frac{1}{2}bh$. Given our desire to build everything from definitions and not be told something that can be figured out, let's embark on an understanding of this formula. Let's determine how we can come up with this formula using only the area formulas for rectangles and parallelograms.

Question 7.2.3. How could we use the area formulas for rectangles or parallelograms to find the area of the triangle below?

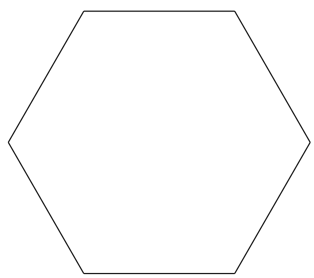


Question 7.2.4. How could we use the area formulas for rectangles or parallelograms to find the area of the triangle below? What information would we need about the triangle in order to compute the area?

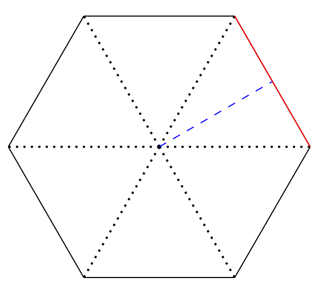


After carefully answering the previous question, we now have an argument to justify why the formula for the area of a triangle is $A = \frac{1}{2}bh$. Therefore we now have at our disposal formulas for the area of rectangles, parallelograms, and triangles. Many figures that we want to find the area of can be broken into these three shapes, and then we can compute the area.

Example 7.2.5. How could we find the area of the regular hexagon shown below? What would we need to know about the hexagon in order to compute the area?



Possible Solution. We could break the hexagon into six triangles (shown below), and since it is a regular hexagon all of the triangles are congruent. Therefore the area of the hexagon would just be six times the area of a single triangle. To find the area of a single triangle we would need to know the length of the side on the hexagon because that will serve as our base for the triangle (shown in red). We would then need to find the distance from the center of the hexagon to the midpoint of a side because that will serve as the height of our triangle (shown in dashed blue).



If we call the length of the red segment b and the length of the dashed blue segment h, then the area of this regular hexagon would be given by $A = 6 \cdot \frac{1}{2}bh = 3bh$. □

With the three formulas we know so far we could find the area of any polygon. That is because polygons have straight edges and so can be broken down into rectangles, parallelograms, and triangles. However, shapes with curves, for example a circle are not so easy. Previously we were able to estimate the area of a circle, but it would be nice to have a formula. To truly derive the formula for the area of a circle requires some very complex mathematics and so we will not be able to show everything needed to understand the formula for the area of a circle. However, there is definitely some progress we can make in trying to understand the formula for the area of a circle.

The piece of mathematical knowledge that we will have to just accept at this point is that the circumference of a circle can be found by computing twice pi multiplied by the radius. In other words, $C = 2\pi r$. (Recall that circumference just means the perimeter of a circle.) We will use this formula to help us understand what the formula for the area of a circle is.

Question 7.2.6. How could we estimate the area of a circle using rectangles or parallelograms or triangles? How could we make our estimate better?

After we have answered the question above, we should consider the pros and cons of the methods we found to estimate the area of a circle. It would be nice if there was a consistent way to increase the accuracy of our estimation. In particular, is there a way to increase the accuracy of our estimate without having to do a great deal of extra work?

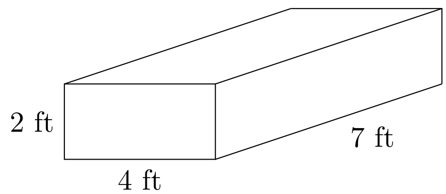
Question 7.2.7. Is there a method of estimation that will allow us to estimate the area of the circle fairly efficiently? How does that method help us better understand the formula for the area of a circle?

The formula for the area of the circle, $A = \pi r^2$, does arise from the estimation method we chose above. It is interesting to note that we remember the formula in the simplified version, namely "pi r squared". However, as we see after answering the question above, the area really arises from computing half the circumference multiplied by the radius. In other words, an alternate form for the area of the circle could be $A = \frac{1}{2} \cdot C \cdot r$. Although I am not advocating that the formula we memorize in school be changed, I do think it useful to see the area of a circle in this light because it gives a little more meaning to a seemingly out of the blue formula.

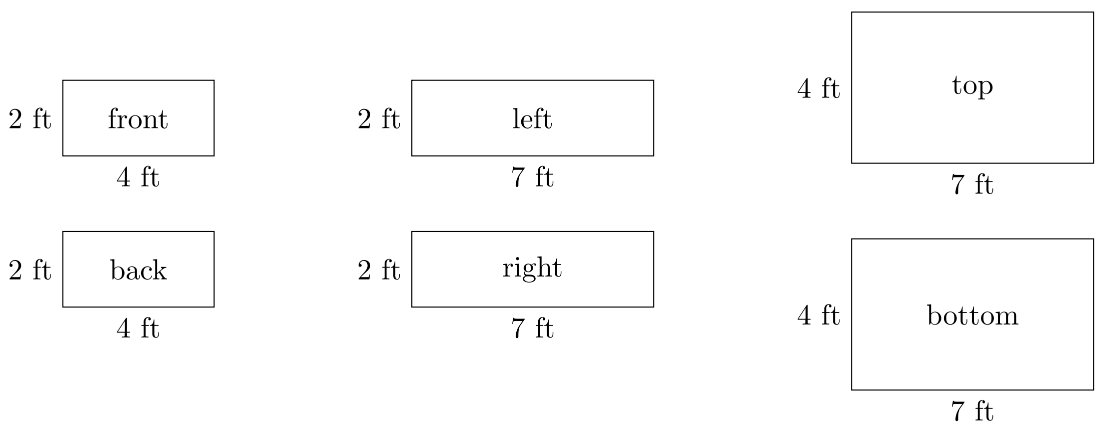
At this stage we are well equipped to find, or least get very good estimates of, the area of any shape by breaking it down into rectangles, triangles, parallelograms, or portions of circles. For many shapes, estimation will have to be good enough for us at this time. For those more unusual shapes, gaining the accuracy of area that we desire would require knowledge of calculus. Obviously we will not be teaching elementary students calculus, but it certainly doesn't hurt for them to hear about interesting mathematical avenues that will be open to them in the future.

7.3 Surface Area

In the previous sections we were working only in two dimensions. In other words, everything we were looking at could be drawn on a piece of paper. We now move into three dimensions. When we were working in two dimensions the notion of area was essentially a measurement of how much fit inside the shape. The way we measured that was by determining how many one unit by one unit squares we could fit in the shape. We do not want to change the notion of area once we get to three dimensions. In other words, we still want area to be a two dimensional concept. However, we can apply the notion of area in a three dimensional setting. When we are answering a two-dimensional area question in a three dimensional setting we often refer to it as **surface area**. What this means is that we are only going to look at the surface of a figure (which is a two-dimensional thing) and find the area of it. This can be more easily understood through an example, so let's consider the three dimensional box shown below.



If we look at the surface of this box we see that it is made up of six rectangles. The rectangles in the front and back are the same, the rectangles on each side are the same, and the rectangles on the top and the bottom are the same. The six rectangles labeled with their location and dimensions are shown below.



When we find the total area of all of these shapes, that is what we refer to as the surface area of the three dimensional figure. Adding those areas up, $8 + 8 + 14 + 14 + 28 + 28 = 100$, we see that the surface area of the box is 100 square feet.

In general, to compute the surface area of a given three dimensional figure we must just determine what two-dimensional shapes create the surface of the figure and use our knowledge of area previously developed to compute the areas of each of those shapes. Consequently we will not spend much time discussing surface area formulas in this section but rather will address most of the topic in the exercises.

There are two notable exceptions to the discussion in the previous paragraph, namely the surface area of a sphere and a cone. Recall that in the previous section we were not able to *completely* explain how to derive the formula for the area of a circle because parts of it required much more advanced mathematics. What we were able to do with the circle however was to approximate it using a bunch of triangles, and we used our knowledge of triangles to find the area of the circle. For the sphere and the cone we can use similar ideas of approximation. In particular, we can approximate spheres and cones using a bunch of shapes and use our knowledge of those shapes to find the area. Although the method is similar, the technicalities of the computations are somewhat more difficult. This difficulty arises from the fact that in the circle the triangles were all congruent where as the shapes we would use to approximate the spheres and cones are only similar to each other not congruent. The difference in the size of the shapes causes a fair amount of difficulty, and is not something that is accessible to elementary students. As a teacher of those students however, you do have enough mathematical background to understand these formulas and should make sure to do so. The best way to really grasp the formulas is through a visual explanation, and youtube has several excellent videos explaining the surface area formulas for spheres and cones, so I will leave that portion of the explanation to youtube.

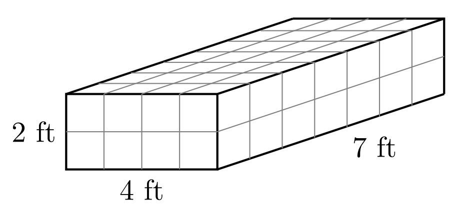
We now state the aforementioned formulas for the surface area of a sphere and a cone.The formula for the surface area of a sphere is $SA = 4\pi r^2$. For the cone*, we need a bit of vocabulary before providing the surface area formula. In a cone the *slant height* is the distance from the edge of the base to its apex. In other words, the distance from the bottom to the top traveling along the outside of the cone. To write down the formula for surface area of a cone we do need to recognize there are really two pieces to the cone. There is the base of the cone, which is just a circle, and then there is the lateral surface. Therefore the formula comes from adding up the area of each of those parts. If we let l represent the slant height, then the surface area formula for a cone is $SA = \pi r^2 + \pi rl$.

*Note: I am using the basic geometry notion of cone, which means when I refer to cone in this book it actually means a right circular cone.

7.4 Volume

Back in two dimensions area was the notion of "how much fit inside", so we still want that notion in three dimensions as well. Consider the box used as an example in the previous section. We would like to have some form of measurement that tells us how much fits *inside the box* because surface area only addressed the faces of the box, not the inside. With area we used 1 unit by 1 unit squares and determined how many of those we needed to fill the shape. We extend this idea to three dimensions and use a 1 unit by 1 unit by 1 unit cube as our measurement tool. Therefore the **volume** of a figure is the number of 1 unit by 1 unit by 1 unit cubes that can fit inside the figure. As with area, the cubes may need to be broken up in order to fit in the shape perfectly.

Using our understanding of volume, let's find the volume of the example box in the previous section. With area we broke the shape into squares and counted them (before we developed formulas), so we can proceed in the same manner here. We need to break our box down into cubes and see how many there are.



Therefore we see that this box has two layers (because of the 2 foot height). In each of those layers there are 4 rows of 7 cubes (because of the 4 foot width and the 7 foot length). Therefore in one layer there are 28 cubes. Since there are two layers, we have a total of $2 \times 28 = 56$ cubes. Thus the volume of this box is 56 cubic feet.

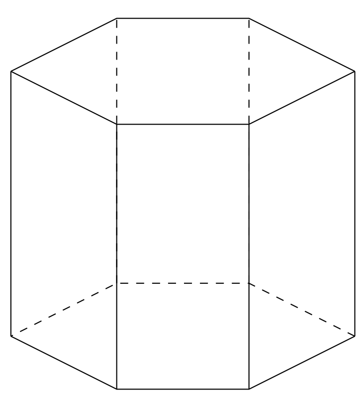
Note that in a three dimensional setting we have two notions of "how big" an object is, namely surface area and volume. It is important to keep in mind that they are measuring two different things. Let's go back to the box example. Knowing the surface area will allow us to understand how much paint would be needed to paint the outside of the box for example. Where as knowing the volume will allow us to understand how much stuff we can put inside the box. Both are important notions of size and are very different.

Question 7.4.1. Is it possible for two different boxes to have the same surface area but different volumes?

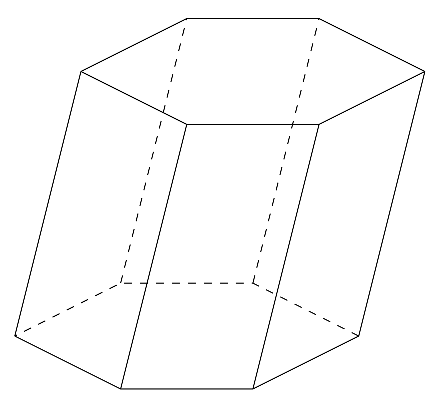
We can now begin developing volume formulas for some common figures. In fact, we have already addressed the most basic of these figures, namely a box, or more precisely a **rectangular prism**. In the example above we merely used our understanding of volume to easily compute the volume of the rectangular prism in question by looking at layers of cubes. A careful analysis of that example allows us to see why the formula for the volume of a rectangular prism is $V = l \times w \times h$.

Question 7.4.2. How can we use our understanding of volume to justify the formula for the volume of a rectangular prism?

The ability to see identical layers in our figure was an extremely useful aspect of our method. Any straight-edged figure that gives rise to this notion of identical layers is referred to as a prism. Let's be a little more precise with the definition of that word. A **prism** is a figure whose base is a polygon and any cross-section of the figure parallel to the base is a polygon congruent to the base. We name different types of prisms by the shape of their base. Note that the definition of prism does not imply the figure must be going straight up from the base; it could be slanted. When we want to refer to a prism that does in fact go straight up, in other words when the sides are perpendicular to the base, we refer to that as a **right prism**. An example of a right prism and a non-right prism are shown below.



Example of a right prism



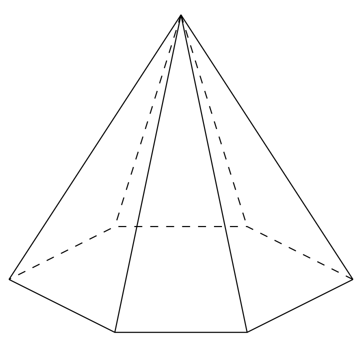
Example of a non-right prism

Finding the volume of a right prism can be achieved by using the layers idea like we did with the box. There will be differences depending on the shapes involved, but the methods will be similar.

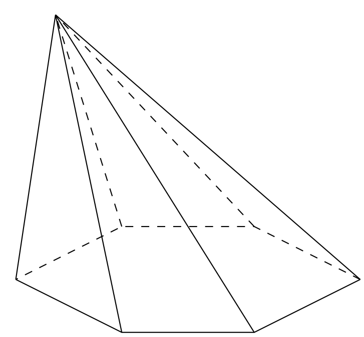
Applications of this method to other right prisms will be explored in the exercises. Given the straight up and down nature of a right prism, it is probably not too much of a stretch to believe that we can deal with them similarly to boxes. What happens though, if we are dealing with a non-right prism? How does the fact that it is slanted affect our methods to find the volume? This too will be addressed in the exercises.

So far we have been able to reason our way through volume, but again we come to the point where some of the explanations require knowledge beyond the elementary level. Consequently this is another spot where you will want to develop this understanding for yourself even though you will not be teaching it to your students. It is worth noting that even though you will not be teaching your students the reasoning behind these formulas they will be expected to learn it later on in their mathematical career, so your understanding of the process will help you decide what pieces you can give them at the level they are at that will help them when they see the complete explanation in the future.

We now come to those figures that fall into this "more difficult reasoning" category. It is probably not a surprise that both spheres and cones again fall into this category. A new culprit however is a pyramid. Let's first be a little more precise about the definition of that word. A **pyramid** is a figure whose base is a polygon and the sides of the figure go up from the base into a point. If the base is a regular polygon and the point is directly above the center of the base we say it is a **right pyramid**.



Example of a right pyramid



Example of a non-right pyramid

We finally state the formulas for the aforementioned figures. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. For a cone or a pyramid, we need to first define what we mean by the height of the cone or pyramid. Draw a line segment that goes through the top of the cone or pyramid and is perpendicular to the base. The length of this segment is the height of the cone or pyramid and is usually denoted by h. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. Finally, the volume of a right pyramid is given by $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height of the pyramid.

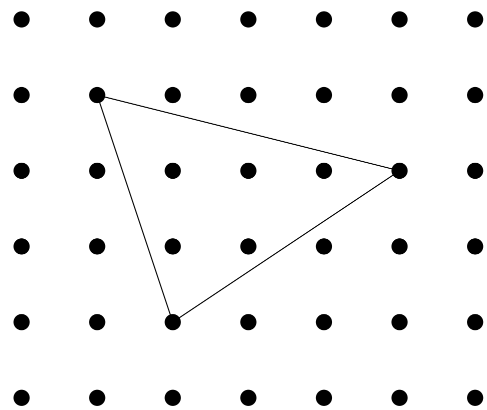
Question 7.4.3. What similarities do you notice about the formula for the cone and the pyramid?

We end with a brief discussion of a similarity that all three formulas have. Notice that they each have a $\frac{1}{3}$ in them. (To see it in the sphere formula we can think of $\frac{4}{3}$ as $4 \cdot \frac{1}{3}$.) When carefully analyzing the method used to develop each of these formulas we will see a similarity in all three methods that give rise to this $\frac{1}{3}$. It is also interesting to look at what is in the rest of the formula when we do pull out that $\frac{1}{3}$.

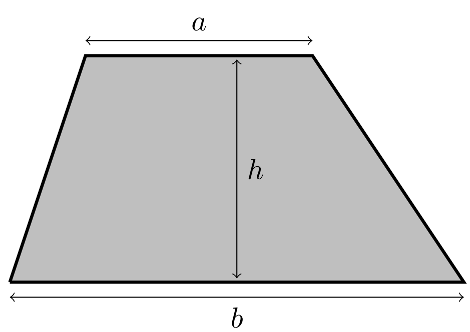
Question 7.4.4. Each of the three formulas above can be viewed as $\frac{1}{3}$ times something. What is the something?

7.5 Exercise

1. On the geoboard shown below, the pegs are one inch apart. Using only the definition of area, find the area of the triangle shown. Explain how you found the area.



2. Consider the trapezoid below.



   (a) Decompose this trapezoid into shapes that we know how to find the area of, i.e. triangles, rectangles, or parallelograms.

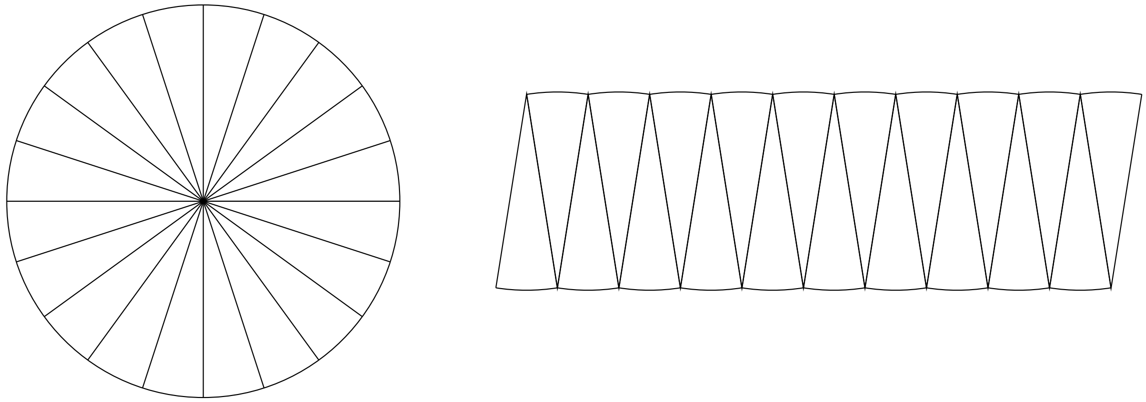
   (b) Use the decomposition you found above to express the area of the trapezoid in terms of a, b, and h.

   (c) Make two copies of this trapezoid and put those two copies together in a way that creates a shape we know how to find the area of.

   (d) Use the shape you created above to express the area of the trapezoid in terms of a, b, and h.

   (e) Are the formulas you found in part 2b and part 2d equivalent?

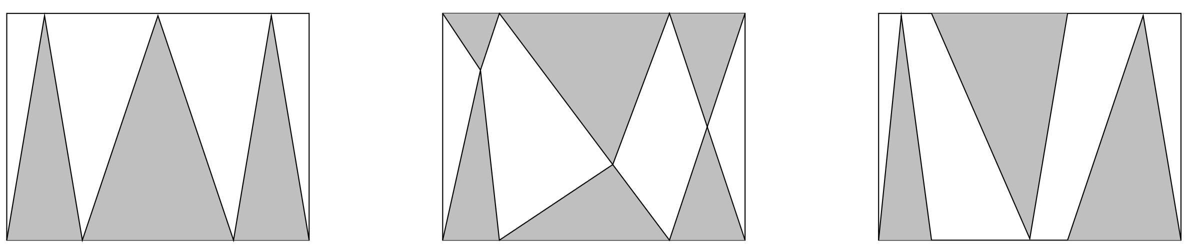
3. A pizza is cut into lots of pieces, and then placed into a rectangular box. Below you can see the pizza as it is cut up, and the way it is rearranged to fit into a rectangular box. You can see that if it is cut into more pieces, it will be a better fit in the rectangular box, i.e. it becomes more of a rectangle.



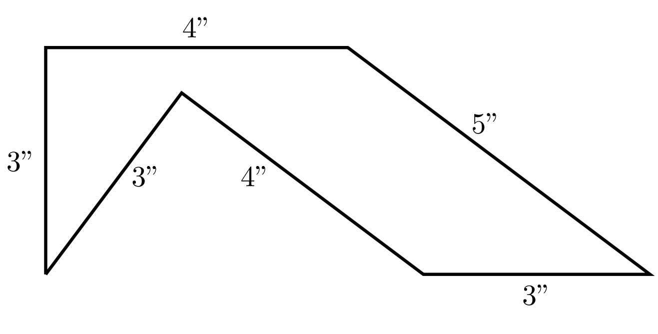
(a) In terms of the radius r and the circumference C of the pizza, what are the approximate dimensions of the rectangular box? Use this to find the area of the rectangular box in terms of C and r.

(b) Now use the relationship discussed in the reading, $C = 2\pi r$, to describe the area of the box in terms of just r (and π), but not C.

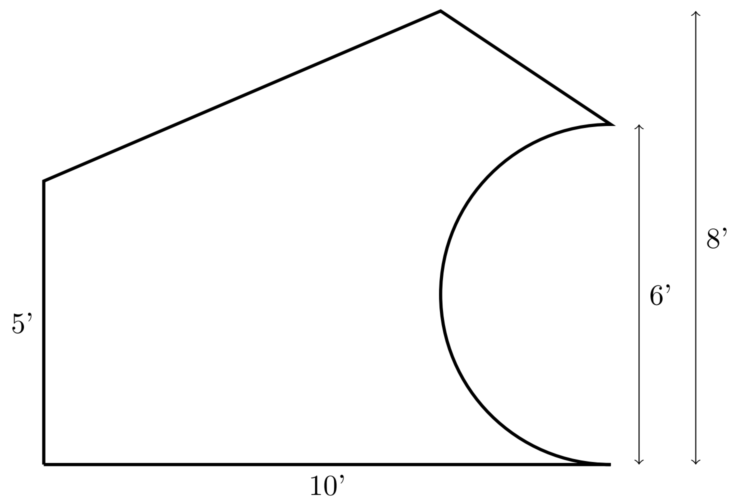
4. The following three rectangles are all the same size. Compare the shaded areas (i.e. greater area/smaller area). Justify your response.



5. Find the area of the figure $ABCDEF$ shown below. Explain how you found the area. (You may assume that angles that look like right angles are in fact right angles.)



6. Find the area of the figure $ABCDEF$ shown below. Explain how you found the area. (You may assume that angles that look like right angles are in fact right angles. You may also assume the point at the top of the figure is directly above the edge of the curved arc.)



7. (a) How could we take a piece of paper and, using tape and scissors, create a cylinder?
   (b) Explain how to use what you did above to determine a formula for the surface area of a cylinder.
   (c) Using only the definition of volume and our knowledge of area formulas, explain why the formula for the volume of a cylinder is $V = \pi r^2 h$.

8. A tetrahedron is a pyramid whose faces are all congruent equilateral triangles. In other words, the base is an equilateral triangle and each side of the pyramid is an equilateral triangle.
   (a) What dimensions would you need in order to find the surface area of a tetrahedron?
   (b) If you want to make a tetrahedron out of balsa wood for your desk and you want to use exactly 2 square feet of wood. What will the dimensions of the tetrahedron be?
   (c) What will the volume of your tetrahedron be?

9. One of your students is looking at two rectangles on their computer screen, but you can’t see the screen. You hear her say, “The first rectangle is twice as big as the second one.”
   (a) Give at least two different interpretations of that statement. In other words, what might she be seeing?
   (b) Are all of your interpretations above equivalent? If so, explain. If not, think of another interpretation of this statement that would not be equivalent.

10. Is the following statement always true, sometimes true, or never true? Justify your response.

    Given any two triangles, the one with the larger perimeter will have a larger area.

11. Suppose you have a rectangle in front of you and a rectangular prism in front of you. Please justify all responses.

(a) If you triple the length and width of the rectangle, how will the area of the new rectangle compare to the area of the original rectangle?

(b) If you triple the length, width, and height of the rectangular prism, how will the surface area of the new rectangular prism compare to the surface area of the original rectangular prism?

(c) If you triple the length, width, and height of the rectangular prism, how will the volume of the new rectangular prism compare to the volume of the original rectangular prism?

12. Consider the following scenario

> You are making a sand pit with a brick border for your garden. The sand pit must be rectangular and the sand must be 2 inches deep. You have 50 bricks whose dimensions are 2 inches by 3 inches by 8 inches, and you have 60 cubic feet of sand.

(a) Children learn the formulas for perimeter, area, and volume, but often do not know how to use them in practice. Indicate how each of these geometric ideas show up in this problem.

(b) Provide labeled sketches of two different sand pits that could be made under these circumstances. For each of your solutions, determine how many bricks were not used and how much sand was not used. Show enough work to support your answers.

(c) The bricks being used are quite expensive so you want to use all of them. Provide a labeled sketch of an acceptable sand pit that uses all of the bricks. How much sand is not used? Show enough work to support your answers.

(Your answer to this question must be different than the two sand pits you found in the previous problem.)

(d) Provide a labeled sketch of a sand pit that could be made with the given amount of bricks, but for which you would not have enough sand. How much more sand would you need? Show enough work to support your answers.

13. Suppose you want to make a rectangular cardboard box with no top by cutting out squares in the corners of a flat piece of cardboard, bending it up to make the sides, and taping them.

(a) If your flat piece of cardboard is 8 feet by 6 feet, give the dimensions of two different size boxes you could make.

(b) Could you make a box with a surface area of 48 square feet? If so, what are the dimensions? If not, why not?

(c) Could you make a box with a surface area of 32 square feet? If so, what are the dimensions? If not, why not?

(d) If you cut a length of 2 feet out of the flat piece, what will the volume of the box be?

(e) If you cut a length of x feet out of the flat piece, give an expression for the volume of the box.

(f) Use a graphing device of some sort and sketch the graph of your volume equation found above. Use the graph to answer the following question: Is it possible to make a box whose volume is 25 cubic feet?

14. The figure below can be folded to form a closed three dimensional figure.

(a) Sketch the resulting three dimensional figure.

(b) What measurements would you need to find the surface area of this figure? If given those measurements, how would they be used to find the surface area?

(c) What measurements would you need to find the volume of this figure? If given those measurements, how would they be used to find the volume?

15. Jaycee wants to make a Mickey Mouse head out of clay. The head will be made with one sphere of diameter two inches and two smaller spheres of diameter one inch. She can buy clay bricks that are 10 cubic inches. How many bricks of clay will she need?

8 Fractions

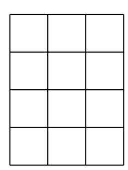
8.1 Building on Previous Knowledge

In previous chapters we covered arithmetic with positive and negative whole numbers. Therefore we understand the four operations: addition, subtraction, multiplication, division. As we begin to study fractions we want to rely on the understanding of these operations that has already been developed. We do not want students to think that these operations are suddenly different because we are working with fractions. In preparation for this, let's review the operations.

When students learn addition in kindergarten they usually learn it as an "all together" idea. For example if they want to compute $2+3$, they put 2 things on the table, then 3 things on the table and count how many they have all together. This is usually the first understanding of addition, but as the students get more advanced they may began to use a number line or to "count on". For example, a student will got to 2 on the number line (or think about 2 in their head) and then count three more spaces on the number line (or count on three more in their head), so you might hear the student say, "two, three, four, five", while keeping track on fingers how many times he/she counted up. For a more in depth review of addition refer to section 2.5.

Subtraction is the inverse of addition. For example, if we want to compute $7-3$, then we need to think, "what can I add to 3 in order to get 7". Though this is the way subtraction is defined, students also often use the "take away" idea to compute a subtraction problem. For example, to compute $7-3$ a student might put seven things on the table, remove three, and then count what's left. As students become more advanced and begin to compute in their head, you may hear them count on from 3, or you may hear them count back from 7. In the former they are using the definition (which we called the missing addend method), and in the latter they are using the take away method. For a more in depth review of subtraction refer to Section 2.6.

When computing a problem like 4×3, students are asked to compute how much four groups of three would be. In other words in a multiplication problem one of the numbers tell us how many are in a group and the other number tells us how many groups. The answer to the multiplication problem is the total amount. A visual used to represent this multiplication problem is shown below.



This visual is also a nice way to see the commutative property of multiplication. Looking at the array above, we see 4 rows of 3, but looking at it in the other direction (turn your head sideways) we see 3 rows of 4. Thus 4 rows of 3 and 3 rows of 4, have the same amount, so $4\times 3=3\times 4$. For a more in depth review of multiplication refer to Section 2.7.

Division is defined to be the inverse of multiplication. For example, to compute $20\div 4$, we need to figure out what to multiply by 4 in order to get 20. Since $4\times 5=20$, then $20\div 4=5$. Keeping in mind however, that 4×5 could mean four groups of five or it could mean five groups of four, then we get two different ways to interpret a division problem. If 4 is the number of groups, then we could interpret $20 \div 4$ in the following way. If we have a total amount of 20 and we fairly make 4 groups, then $20 \div 4$ is the amount in one of those groups. We called this method division by sharing. Alternately, if 4 is the amount in a group, then $20 \div 4$ could be represented by the following. If we have a total amount of 20 and we make groups of size 4, then $20 \div 4$ is the number of groups we could make. We called this method division by grouping. For a more in depth review of division refer to Section 2.10.

Though we define each of the operations in a specific way, there is more than one way students can think about those operations when computing. We would like our students to be comfortable using whichever method suits the problem best. For example, if a student is trying to compute $35 - 2$ mentally, we would hope that they would just count backwards 2 from 35. In other words, we would hope that they would use the take away method of subtraction rather than the missing addend method. However, if they are computing $26 - 23$ in their head, then we would hope they would count up 3 from 23, so in this case the missing addend method is more useful. As discussed previously, we want to encourage flexible use of numbers and operations, so giving students opportunities to decide what method is useful to them in which settings is an important piece of their development. This flexibility continues to be important with fractions as well. In particular, students often don't think of fractions as actual numbers but rather some abstract construct. The more we can get students to think about fractions in the same way they think about whole numbers the better off they will be.

As we embark on our exploration of fractions we want to keep our current understanding at the forefront and try to build everything we know about arithmetic with fractions on our previous knowledge of the four operations and on the definition of a fraction. We do not want our students to think things are suddenly different because we are dealing with fractions. For example, addition and multiplication are commutative; that doesn't change just because we are working with fractions. Therefore all of the properties we know about these four operations still hold. Our goal throughout our study of fractions (actually throughout the whole course) is to develop a logical progression from previous knowledge and new definitions to an *understanding* of the task at hand. There should never be a need to just state a rule to memorize.

8.2 Definition of a Fraction

Think about how you would explain to a young child what $\frac{1}{2}$ means. For example, you may say take a cookie and break it into two equal pieces and one of those pieces would be half of a cookie. We will use this idea to define a fraction, but first let's look at an example.

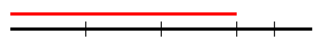
Example 8.2.1. The black rectangle represents one whole candy bar. How much of the candy bar is shaded?

Answer. The whole candy bar is made up of 9 pieces, and the shaded part is taking up 7 of those pieces. Thus the shaded part is $\frac{7}{9}$ of the candy bar. □

Example 8.2.2. If the black segment below is 1 foot long, then how long is the red segment?

Answer. The 1 foot segment is broken into 4 equal pieces and the red segment takes up 3 of those pieces. Therefore the red segment is $\frac{3}{4}$ of a foot long. ☐

Question 8.2.3. A student drew the following picture for $\frac{3}{5}$. Why doesn't the red shading below represent $\frac{3}{5}$ of the line segment?



Now that we have looked at a few examples and gotten our heads back into fractions, we can begin to think about where fractions really begin. In other words, what is the definition of this strange looking thing, $\frac{a}{b}$? From the examples above we can pull out the important aspects of a fraction. First of all, we had to declare what we were starting with. For example, 1 candy bar or 1 foot long segment. In other words, what is 1 in our setting? Now this might at first glance seem like this is a new thing to have to do, but we actually did this with whole numbers as well. If we think about what the number 2 means, it means 1 and another 1. Thus even whole numbers are defined in terms of what we choose to be 1.

When we choose what 1 is when dealing with fractions we will call this **designating a whole**. Once we have designated the whole, we can then describe what role the numbers on the top and bottom play. We are now able to give the precise definition of a fraction. We will take this definition, and our previously developed knowledge, to build a complete understanding of fractions.

Definition 8.2.4. *Definition of* $\dfrac{a}{b}$

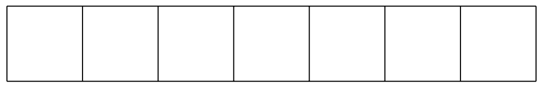
1. Designate a whole.
2. Break the whole into b equal pieces.
3. $\dfrac{a}{b}$ is represented by a pieces.

Let's take some time to look at examples with different wholes. Students usually like pizza and candy bars so that gives us two types wholes to use, namely a circle (pizza) and a rectangle (candy bar). Using circles as a whole is very common, but it does have its draw backs. If you are drawing halves or fourths or eights, etc., then it is not too hard to get equal pieces. However, if you are trying to draw thirds, fifths, sevenths, etc., then getting equal pieces on a circle can prove to be difficult.

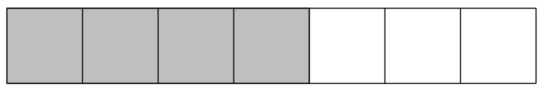
Example 8.2.5. Use a candy bar to represent $\frac{4}{7}$.

Possible Solution. We must begin by designating our whole. In other words, what will one whole candy bar look like? Our choice for one whole candy bar is shown below.

Now, according to the definition of a fraction, we need to break this one whole candy bar into 7 equal pieces.



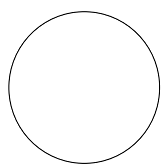
Again, according to the definition of a fraction, we need to use 4 of those pieces.



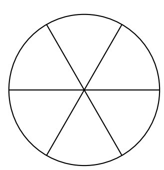
The shaded region represents $\frac{4}{7}$. $\square$

Example 8.2.6. Use a pizza to represent $\frac{5}{6}$.

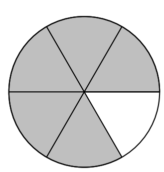
Possible Solution. We begin by designating what one whole pizza is.



We now need to break this whole pizza into 6 equal pieces.



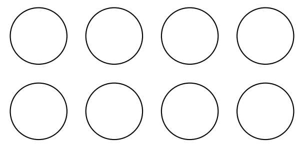
Finally we need to shade 5 of those pieces.



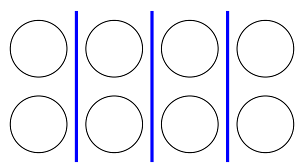
Therefore the shaded portion represents $\frac{5}{6}$. $\square$

In all of the examples we have done so far our whole was one continuous piece. However, a whole can be anything. When the whole is not one continuous piece, we call that a discrete model, and an example is shown below. In the following example, the whole is the group of 8 circles.

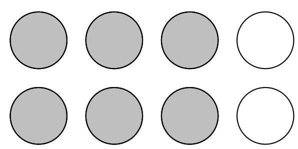
Example 8.2.7. Use the group of coins below to represent $\frac{3}{4}$.



Possible Solution. In this case, our whole is this *one group* of coins. We need to break this group of coins into 4 equal pieces. This is shown with the blue lines below.



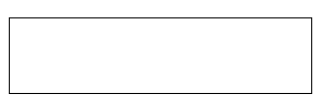
We now want to shade the circles in 3 of those pieces.



Therefore the shaded portion represents $\frac{3}{4}$ □.

Needless to say students will need to do lots of examples to really own their understanding of fractions. One good way to really push their thinking is to ask them "backwards questions". If you notice in all of the problems above we were giving the whole and were asked something about the fraction. That is how most questions are stated about fractions, especially to begin with. However, if a student really understands the definition of a fraction we shouldn't have to always ask the questions in the same way. For example, rather than being given the whole and asking a question about the fraction, we could give the fraction and ask something about the whole. The example below is what I would consider a backwards question in this topic.

Question 8.2.8. If $\frac{3}{5}$ of a candy bar is shown below, then how big was the whole candy bar?



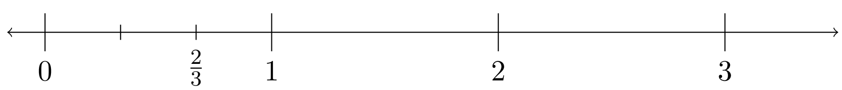
It is sometimes helpful for students to have hands on activities that allow them to play with the definition of fractions. Some common classroom tools are fraction strips, pattern blocks, and

Cuisenaire Rods. Googling any one of these three things will bring up all kinds of activities related to fractions.

In all of our examples so far, there is one glaring omission! The number line! If we want students to be grounded in the fact that fractions are indeed numbers, then they had better be comfortable working with fractions on the number line. The nice thing about a number line is that the whole has already been designated. In other words, 1 has to be on the number line somewhere and once we know where 1 is, we can place any fraction on the number line. Recall that we label a spot on the number line by its distance from 0. So, for example, if we want to mark where $\frac{2}{3}$ is on the number line, we want to mark the spot that is a distance of $\frac{2}{3}$ from 0. If we know what a distance of 1 is, then we can apply the definition of a fraction to place $\frac{2}{3}$. This is shown in the example below.

Example 8.2.9. Mark $\frac{2}{3}$ on the number line below.

Possible Solution. We must begin with a length of 1, so we will consider the segment from 0 to 1. We need to break that segment into 3 equal pieces. Then a segment that takes up 2 of those pieces will have a length of $\frac{2}{3}$. This is shown below.



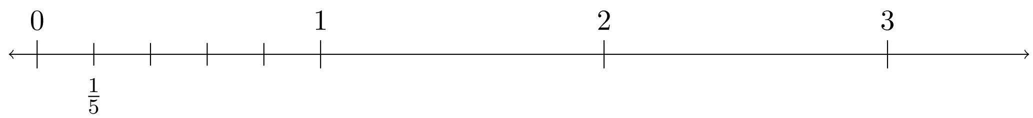
□

Students who are able to work with fractions on a number line tend to have a better grounding in fractions. In addition, if students are able to just live in the definition of a fraction without any "rules" for a while, then they have a better chance of understanding throughout the topic of fractions. For this reason we will spend the next section working on the number line and exploring what we can do with our previous knowledge and this new definition.

8.3 Fractions on the Number Line

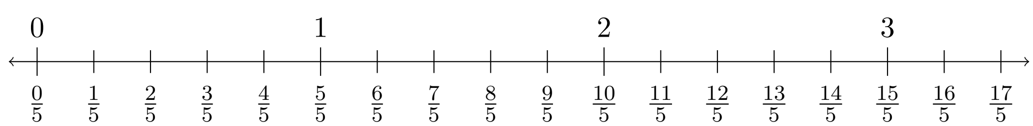
One of the issues with fractions that tends to cause problems is the fact that a number can have more than one name. For example, start with the number line below.

Let's now mark $\frac{1}{5}$ on this number line. So we need to break the segment from 0 to 1 into five equal pieces.



According to the definition of a fraction we have created the pieces we need to represent $\frac{1}{5}$. Of course, we have also created the pieces we need to represent $\frac{2}{5}$ as well. All that means is that we need two of those pieces. What if we take five of those pieces? That would be the fraction $\frac{5}{5}$. But if we really do move along the number line 5 pieces from 0, where do we end up? Yep, the number 1. Therefore 1 and $\frac{5}{5}$ are the same number.

Notice that there is also nothing stopping us from continuing those tick marks. In other words continue to mark off a length of $\frac{1}{5}$. So we are now able to make sense of numbers like $\frac{6}{5}$ or $\frac{12}{5}$. This type of fraction, when the numerator is greater than or equal to the denominator, is called an **improper fraction**. By the way, I think that is a terrible name! There is nothing improper about them. In fact, we will see that fractions written in this way can be extremely useful. I have a friend who refuses to call fractions improper and instead always refers to them as "useful fractions". We have completed the number line below with these useful fractions.



Example 8.3.1. What if our number line went out to 8? How many fifths would that be?

Possible Solution. We know that a segment of length one has 5 pieces. If we go out to 8, then we would have 8 segments of length 1. In other words, we would have 8 groups of 5 pieces. Thus we would have 8×5 or 40 pieces. Therefore when we reach 8 on the number line, we would have 40 fifths. In other words, $8 = \frac{40}{5}$. □

Example 8.3.2. What whole number is equal to $\frac{35}{5}$?

Possible Solution. We know that that a segment of length one has 5 pieces, so we need to figure out how many of those segments we have. If we have a total of 35 pieces, and we divide those 35 pieces into groups of 5, then we will have 7 groups $(35 \div 5 = 7)$. In other words we can make 7 segments of length 1. Thus $\frac{35}{5} = 7$. □

Example 8.3.3. Between what two whole numbers will $\frac{32}{5}$ land on the number line? How far past the first number will it be?

Possible Solution. Since we are working with fifths, we know a segment of length 1 is broken into five pieces. We have 32 of those pieces, so we need to figure out how many groups of five we can make with those 32 pieces. Since $32 \div 5 = 6$ r. 2, then that means we can make 6 groups of 5 pieces and we will have 2 pieces left over. Therefore $\frac{32}{5}$ is past 6, and we don't have enough pieces to reach all the way to 7, so $\frac{32}{5}$ is between 6 and 7. Moreover, we know we are 2 pieces past 6 and the size of those pieces is fifths, so we are a distance of $\frac{2}{5}$ past 6. □.

This last example gives us another way to describe a fractional number greater than 1. We described $\frac{32}{5}$ as the last whole number it passes and how far past that whole number we went. We would represent this as $6\frac{2}{5}$. So this means 6 wholes and $\frac{2}{5}$ of another whole. When we represent a number in this way we call it a **mixed number**.

Example 8.3.4. What would the improper fraction label be on the tick marked $2\frac{3}{7}$?

Possible Solution. If we were to fill in the segments from 0 to 1 and 1 to 2 with sevenths, then we would have two groups of 7 pieces between 0 and 2. We then have 3 more pieces to get to the red tick mark. So in total we would need 17 pieces. The size of those pieces are sevenths, so the red tick mark would be $\frac{17}{7}$. □

We now move from just placing a single fraction on the number line to comparing two fractions. Again, we need to be sure we are only using our understanding of the definition of fractions when comparing two fractions. Some comparisons follow directly from the definition.

Question 8.3.5. Which is bigger, $\frac{2}{5}$ or $\frac{4}{5}$?

Question 8.3.6. Which is bigger, $\frac{2}{5}$ or $\frac{2}{9}$?

The definition of a fraction tells us the role each number in the fraction plays and from there we have two things of importance when comparing fractions. How many pieces represent the fraction, and how big are those pieces? Answering the above two questions relies solely on those two ideas.

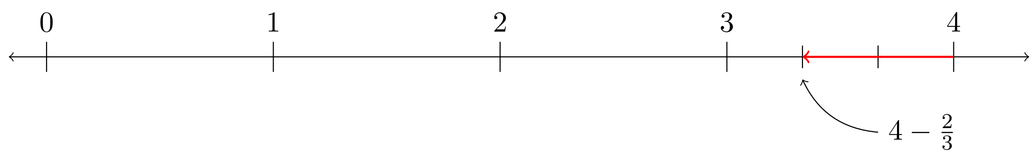
Moving beyond just this basic application of the definition we can actually reason out some more complicated comparisons. For each pair of fractions below, we can determine which is bigger using only what we know about fractions so far. Before moving on, take the time to think through these problems so that you believe these can in fact be reasoned through with what we have so far.

$$\frac{3}{7} \text{ or } \frac{5}{9} \qquad \frac{3}{7} \text{ or } \frac{4}{9} \qquad \frac{7}{8} \text{ or } \frac{8}{9}$$

Let's now move from comparing fractions to operating with fractions. As discussed in the first section of this chapter we want to rely on our already built understanding of operations and apply that understanding to fractions. To reiterate then, whatever we decide to do with fractions should come from what we did with whole numbers.

Example 8.3.7. Compute $4 - \frac{2}{3}$.

Possible Solution. Let's compute this difference on the number line using the take away method. Therefore we want to go to 4 on the number line and then go backwards a distance of $\frac{2}{3}$. In order to do that we need to know how big a third is, so we need to take a segment of length 1 and break it into 3 equal pieces. Because we want to compute $4 - \frac{2}{3}$ we will be going to the left of 4, so it makes sense to break the segment between 3 and 4 into three equal pieces. We can then see how far to go back.



We can see that after moving back a distance of $\frac{2}{3}$ we land at the first tick mark after 3. In other words, one piece beyond 3. Since the size of those pieces are thirds, then we land $\frac{1}{3}$ past 3. Therefore $4 - \frac{2}{3} = 3\frac{1}{3}$. □

Notice that when solving the problem above all we used was our knowledge of subtraction in whole numbers and our knowledge of fractions on the number line (which arose out of our understanding of the definition of a fraction). Our basic understanding of operations in whole numbers will take us far in fractions. Let's consider the division problems $3 \div \frac{1}{4}$ and $\frac{2}{3} \div 4$. First of all, we have two ways of thinking about division, namely division by grouping or division by sharing.

Question 8.3.8. If we are thinking of $3 \div \frac{1}{4}$ as division by grouping, what would a solution look like?

Question 8.3.9. If we are thinking of $3 \div \frac{1}{4}$ as division by sharing, what would a solution look like?

Question 8.3.10. To compute $\frac{2}{3} \div 4$, do you prefer division by grouping or division by sharing?

In answering these questions, we can see that just understanding division in whole numbers gives us a way to compute division problems involving fractions. In a similar manner, each of the arithmetic problems below can be solved using direct applications of our understanding of operations in whole numbers. You are encouraged to really think through these problems to see how a student with only the knowledge we've covered so far could answer them.

$\frac{2}{9} + \frac{5}{9}$	$1\frac{2}{7} - \frac{4}{7}$	$3 \times \frac{2}{5}$
$4 \div \frac{2}{3}$	$\frac{12}{13} \div 3$	$\frac{15}{4} \div \frac{3}{4}$

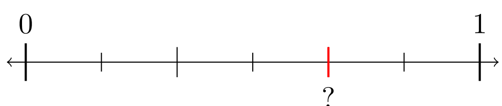
Again, I implore you to work on these problems before moving on to the next section. It is important to see how much can be done without ever needing to develop any "rules". Because, alas, that is where we are headed. We definitely want students to understand fractions deeply, but we also want them to be efficient working with them, so looking for shortcuts is not a bad pursuit as long as students first understand fractions themselves.

8.4 Equivalent Fractions

In the previous section we saw that there was more than one way to represent a whole number. For example, we know that $1 = \frac{5}{5}$ and $1 = \frac{7}{7}$. In other words, 1, $\frac{5}{5}$, and $\frac{7}{7}$ are really all the same number. The notion that two fractions can be equal even though they don't look the same can be difficult for students to grasp. When two different fractions represent the same number, we say they are **equivalent fractions**. In some sense, this wording is misleading. For example, $\frac{5}{5}$ and $\frac{7}{7}$ are equivalent fractions because they both represent the number 1. But if they are both equal to 1, then they are equal to each other, so equivalent fractions are in fact equal. In other words, $\frac{5}{5} = \frac{7}{7}$.

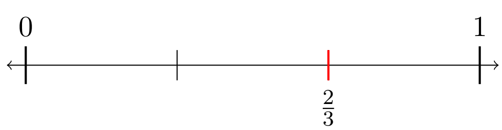
Whole numbers are not the only numbers that can be represented in more than one way. In fact, any number can be represented in more than one way. Consider the fraction $\frac{2}{3}$ placed on the number line below.

The spot marked in red represents $\frac{2}{3}$ on the number line. If we now break each of those thirds into two equal pieces, what fraction does the red mark now represent?

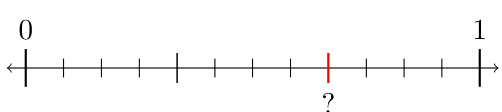


There are now six pieces in a segment of length 1, so the size of those pieces are sixths. The red mark is located four of those pieces from 0. Therefore the red mark is represented by the fraction $\frac{4}{6}$. So both $\frac{2}{3}$ and $\frac{4}{6}$ land in the same spot on the number line. In other words, they both represent the same number. Thus $\frac{2}{3} = \frac{4}{6}$.

Example 8.4.1. The red mark below is labeled $\frac{2}{3}$. If we break each of the thirds into 4 equal pieces, what alternate labeling for the red mark would we have produced?



Possible Solution. Breaking each third into four equal pieces, we get the following picture.



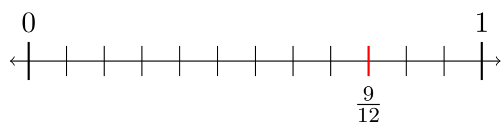
We now have 12 pieces in a segment of length 1, and the red mark is 8 of those pieces from 0. Thus the red mark can also be labeled $\frac{8}{12}$. □

Question 8.4.2. If we instead broke each of the thirds into 20 pieces in the previous example, what would be the new label for the red mark?

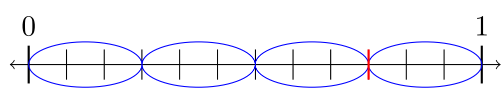
Question 8.4.3. If we started with a picture of $\frac{2}{3}$ and altered it so that it now represents $\frac{16}{24}$, how was the pictured altered?

In the previous examples, we created equivalent fractions by breaking the pieces we started with down into smaller pieces. However, we could also do the reverse. If we have a bunch of pieces to start with, we could glue pieces together to form new, bigger pieces. We explore this idea in the next example.

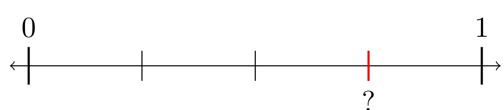
Example 8.4.4. The red mark is currently labeled $\frac{9}{12}$. If we glue every three pieces together, what new label have we created?



Possible Solution. Let's first visualize gluing every three pieces together. We have ringed pieces that will be glued together.



Once we have done the gluing, our picture then looks like this.

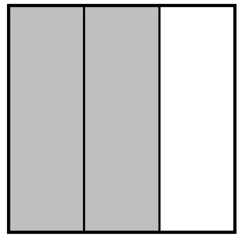


There the red mark could now be labeled $\frac{3}{4}$. □

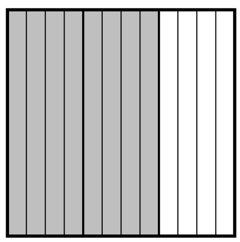
Question 8.4.5. Suppose a number line shows $\frac{25}{35}$ represented. If we glue every five pieces together, what new label will we have produced for that spot?

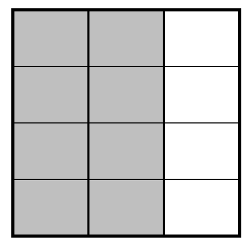
Question 8.4.6. Suppose a number line shows $\frac{18}{24}$ represented. The number line is then altered to so that the new label is $\frac{9}{12}$. How was the number line altered?

We have found that altering the picture that represents a fraction gives rise to an equivalent fraction. We have been working solely on the number line but this method works no matter what your chosen whole is. For example, suppose we want to use the square below as our whole, then the shaded region represents $\frac{2}{3}$.



Let's now break each of those pieces into 4 equal pieces. Notice that we have a choice how to break them up, either vertically or horizontally. We will show both.





In either case we now have 12 equal pieces in the square (our whole) and 8 of the pieces are shaded. Therefore now the shaded region is represented by $\frac{8}{12}$.

Although I personally am partial to the number line, it is important to work in other settings as well because we want to keep that flexibility in our problem solving. No matter what model we use to represent equivalent fractions the arithmetic that arises from our work is the same. At this point we have done enough examples to justify our first shortcut. We should be able to generalize what arithmetic was done to create equivalent fractions, and therefore be able to justify the following.

Property 8.4.7 (Equivalent Fractions)**.** For any non-zero whole number n and fraction $\frac{a}{b}$, we have the following.

$$\frac{a}{b} = \frac{a \times n}{b \times n} \qquad \qquad \frac{a}{b} = \frac{a \div n}{b \div n}$$

We pause here to highlight the fact that with the introduction of the definition of a fraction we have defined a whole new set of numbers. Starting with whole numbers a and b, we have given meaning to numbers of the form $\frac{a}{b}$. We will see in Section 8.8 that we can actually make sense of $\frac{a}{b}$ even if a and b are negative numbers. The problem we have found in the explorations above is that there is more than one way to represent the same number. Therefore we need to be careful how we describe this new set of numbers. We will explore this idea in more depth in a later section, but for now let's at least define this new set of numbers. The set of **rational numbers** is the set of all numbers that can be expressed in the form $\frac{a}{b}$ where a is an integer and b is a non-zero integer.

Let's consider how we can apply this definition. First of all is $\frac{6}{8}$ a rational number? Well, yes because 6 and 8 are integers, so it fits the form of a rational number. What about the number 3? The way it is expressed right now, it is not of the form integer over integer. However, is it possible to express the number 3 in a different way? You can probably come up with lots of ways considering our explorations above. For example, using the definition of a fraction we know that $\frac{3}{1}$ is equal to 3. Also, $\frac{6}{2}$ is equal to 3, etc. So that means 3 is a number that can be written in the form $\frac{a}{b}$ where a and b are integers. Therefore 3 is a rational number. In this way we see that all whole numbers are in fact rational numbers as well.

8.5 Applications of Equivalent Fractions

The Equivalent Fractions Property will allow us to work with fractions much more efficiently. In this section we will develop some of the standard algorithms for fractions. Whenever we are developing these standard algorithms we want to remember not to get lost in the routine and still think about fractions from a conceptual point of view. Recall that just because they are called standard algorithms does not necessarily mean they are the best way to attack every problem.

Let's first explore how the Equivalent Fractions Property gives us a way to compare any two fractions. Rather than go directly to fractions however, let's think about money for a minute instead. Suppose Max has 37 quarters and Macy has 37 dimes. Who has more money? We don't really need to think too hard about this question because quarters are worth more than dimes, and since we have the same number of quarters as dimes, then Max has more money. The reason this problem was easy is because we had the same number of each coin so we need only look at how much the coins were worth. On the other hand, suppose Max has 37 quarters and Macy has 25 quarters. Who has more money now? In this case the coins are worth the same, so we need only figure out who has more coins. Thus Max has more money in this case.

With this money example we see the two things that mattered were how much each coin is worth and how many coins we have. Similarly with fractions we have two things of interest, namely the size of the pieces and how many of the pieces we have.

Question 8.5.1. What would be an example of a fraction comparison problem that would be analogous to having 37 quarters and 37 dimes?

Question 8.5.2. What would be an example of a fraction comparison problem that would be analogous to having 37 quarters and 25 quarters?

You probably remember in school being told to find a common denominator. But why? What are we really doing when we do that? Well, let's go back to the definition of a fraction. The denominator tells us the number of pieces in a whole, so the denominator determines the size of the pieces. If we have the same size pieces then we will be able to easily compare the two fractions.

From our money problem above, we see there is an alternate method. Rather than have the pieces be the same size we could have the number of pieces we have be the same. In other words, why don't we learn about getting a common numerator in school? It is just as useful as common denominator when comparing fractions. Let's look at the same example twice, so we can apply both methods.

Example 8.5.3. Which fraction is bigger, $\frac{7}{11}$ or $\frac{8}{13}$?

Possible Solution. Let's work to get the fractions to have the same size pieces. In other words, we need to find a common denominator. Using the Equivalent Fraction Property, we know that we can multiply the top and bottom by any number and the fraction will still be equivalent to the original. Our denominators are 11 and 13, so we need to find a common multiple of 11 and 13. We can use multiplier 13 for the first fraction and multiplier 11 for the second fraction. This work is shown below.

$$\frac{7}{11} = \frac{7 \times 13}{11 \times 13} = \frac{91}{143}$$
$$\frac{8}{13} = \frac{8 \times 11}{13 \times 11} = \frac{88}{143}$$

Now each of our fractions are represented by the same size pieces, namely one hundred forty thirds. The fraction $\frac{7}{11}$ has 91 of those pieces and the fraction $\frac{8}{13}$ has 88 of those pieces. Therefore the fraction $\frac{7}{11}$ is larger than $\frac{8}{13}$. □

Example 8.5.4. Which fraction is bigger, $\frac{7}{11}$ or $\frac{8}{13}$?

Possible Solution. This time let's work to get the same number of pieces. In other words, we need to get a common numerator. We show this work below.

$$\frac{7}{11} = \frac{7 \times 8}{11 \times 8} = \frac{56}{88}$$
$$\frac{8}{13} = \frac{8 \times 7}{13 \times 7} = \frac{56}{91}$$

Now each fraction is represented by the same number of pieces. However, the fraction $\frac{8}{13}$ is now represented by pieces of size 91sts, which are smaller than 88ths. Therefore we have the same number of pieces in each, namely 56, but the pieces themselves are smaller for the fraction equivalent to $\frac{8}{13}$. Therefore $\frac{7}{11}$ is larger than $\frac{8}{13}$. □

Whenever we are using the Equivalent Fractions Property, notice the important of understanding common multiples. We must be comfortable finding common multiples of two numbers in order to find common denominators or numerators. Notice that as long as we can find *any* common numerator or denominator, we can easily compare the fractions. In particular, we do not have to find the *least* common denominator. For example, when comparing $\frac{3}{4}$ and $\frac{5}{6}$ we could just as easily use a denominator of 24. We do not have to use denominator 12, which would be the least common denominator.

We see that our ability to find a common denominator or a common numerator using the Equivalent Fractions Property allows us to compare *any* two fractions because given any two numbers we can always find a common multiple.

Question 8.5.5. Suppose you have the fractions $\frac{a}{b}$ and $\frac{c}{d}$. What could be used as a common denominator? What could be used as a common numerator?

We now move on to our next application of the Equivalent Fractions Property. Another place where having the same size pieces is important is when adding and subtracting. Let's think about addition of whole numbers and apply those ideas to fractions. When adding whole numbers we can think of it as "all together" or if we are working on the number line we can think of moving along the number line. Let's consider both of these methods with the addition problem $\frac{2}{7} + \frac{3}{7}$.

Using the all together idea we could proceed as follows. We know that $\frac{2}{7}$ is 2 pieces of size sevenths and $\frac{3}{7}$ is 3 pieces of size sevenths. Therefore all together we have 5 pieces of size sevenths. Thus

$\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$. Now let's try the same sum on the number line. We begin by putting our pencil on $\frac{2}{7}$, then move to the right a distance of $\frac{3}{7}$. Once we do that we will end up at $\frac{5}{7}$.

Notice that both of these methods are essentially just counting up the number of pieces because the pieces were all the same size. Since the numerator tells us the number of pieces, we can formally write down how to add two fractions. However, our method relied on the fact that the pieces were the same size. Thus our denominators must be the same.

Property 8.5.6 (Addition of Fractions Property)**.** For any fractions $\frac{a}{c}$ and $\frac{b}{c}$, we have the following.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

The moral of the story here is that if the pieces are the same size then addition is easy. The obvious question here is what happens when the pieces are not the same size? Well, that is where the Equivalent Fractions Property becomes relevant. We know that we can take any two fractions, and find a common denominator, thus making all the pieces the same size.

As we proceed we will gradually be using more and more properties, so as we complete problems we should be able to justify every step. The only difference now is that some justifications will be quite short because we can just state a property.

Example 8.5.7. Find the sum $\frac{4}{7} + \frac{4}{5}$.

Possible Solution.

$$\frac{4}{7} + \frac{4}{5}$$

Step 1: Equivalent Fractions Property.

$$= \frac{4 \times 5}{7 \times 5} + \frac{4 \times 7}{5 \times 7}$$

Step 2: Multiplication of whole numbers.

$$= \frac{20}{35} + \frac{28}{35}$$

Step 3: Addition of Fractions Property.

$$= \frac{48}{35}$$

Thus $\frac{4}{7} + \frac{4}{5} = \frac{48}{35}$. □

Now let's think about how subtraction works. When kids first learn subtraction they normally use the "take away" method. For example, to compute $7 - 3$, they would get 7 things and take away 3, leaving them with 4. We can use this same idea with fractions because we know how to get all the pieces the same size.

Example 8.5.8. Compute the difference $\frac{5}{9} - \frac{2}{9}$.

$\frac{5}{9}$ is 5 ninths and $\frac{2}{9}$ is 2 ninths, so if we take away 2 ninths from the 5 ninths we are left with 3 ninths. Thus $\frac{5}{9} - \frac{2}{9} = \frac{3}{9}$. □

Thus we see just as with addition, as long as we have a common denominator, it is just a matter of how many pieces we have.

Property 8.5.9 (Subtraction of Fractions Property)**.** For any fractions $\frac{a}{c}$ and $\frac{b}{c}$, we have the following.

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Again, as we proceed now, we will use the shortcuts we develop, but we want to be sure to justify each equality so that we are careful not to use anything we haven't reasoned through yet.

Example 8.5.10. Compute the difference $\frac{5}{7} - \frac{2}{3}$.

Possible Solution.

$$\begin{aligned} & \frac{5}{7} - \frac{2}{3} & \\ & & \text{Step 1: Equivalent Fractions Property.} \\ = \; & \frac{5 \times 3}{7 \times 3} - \frac{2 \times 7}{3 \times 7} & \\ & & \text{Step 2: Multiplication of whole numbers.} \\ = \; & \frac{15}{21} - \frac{14}{21} & \\ & & \text{Step 3: Subtraction of Fractions Property} \\ = \; & \frac{29}{21} & \end{aligned}$$

Therefore $\frac{5}{7} - \frac{2}{3} = \frac{29}{21}$. □

8.6 Multiplication

In Section 8.3 we were able to compute some products using our understanding of multiplication. For example $3 \times \frac{2}{5}$ means 3 groups and in each group there is $\frac{2}{5}$. In other words, $3 \times \frac{2}{5}$ is just $\frac{2}{5} + \frac{2}{5} + \frac{2}{5}$, and since we know how to add fractions, we get $\frac{6}{5}$.

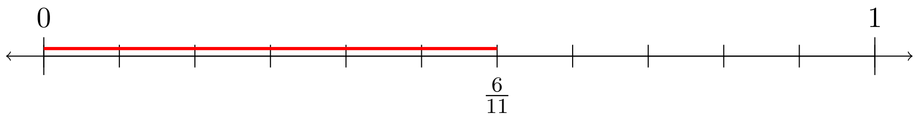
Using the idea above then, we can compute the product of a whole number and a fraction. So then the obvious next question is, how do we compute the product of two fractions. Well, whatever we answer had better rely on our knowledge of multiplication and our knowledge of fractions. Let's be careful with our language so that we can generalize to fractions. In a multiplication problem like $a \times b$, the a tells us how how many b's we want to count. For example, in $3 \times \frac{2}{5}$, we interpreted this as "three" "two fifths".

Question 8.6.1. How could we use this interpretation to compute $\frac{1}{2} \times \frac{2}{5}$?

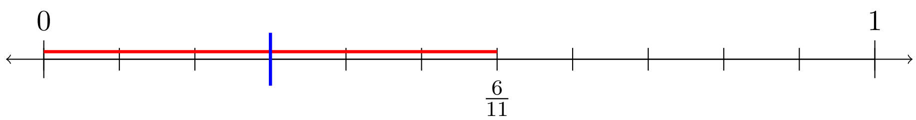
After answering the above question, we notice that in that specific example everything worked out pretty nicely. In particular, $\frac{2}{5}$ was a fraction that we could easily find a half of. Let's take a look at another example.

Example 8.6.2. Compute $\frac{1}{2} \times \frac{6}{11}$.

Possible Solution. We can interpret this has computing half of $\frac{6}{11}$, so let's first get $\frac{6}{11}$ on a number line.



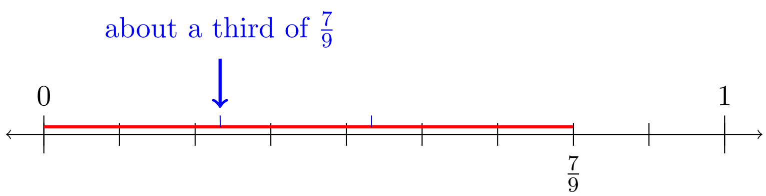
The red segment represents $\frac{6}{11}$. We want to find half of that red segment. The halfway mark is indicated in blue below.



We see that half way to $\frac{6}{11}$ would be the blue mark, which is $\frac{3}{11}$. Therefore $\frac{1}{2} \times \frac{6}{11} = \frac{3}{11}$. □

Again, this was a multiplication problem that worked out nicely. We could probably think of lots more that would could be worked out pretty easily. For example, $\frac{1}{3} \times \frac{12}{17}$ or $\frac{1}{4} \times \frac{8}{15}$ or $\frac{1}{5} \times \frac{15}{19}$. Before moving forward be sure you are comfortable finding each of these products.

But let's now consider the problem $\frac{1}{3} \times \frac{7}{9}$. If we try what we have done above it will not work out as nicely, but that doesn't mean the method doesn't work. We still should be thinking about how to find a third of $\frac{7}{9}$. We can see below that we have an idea of about where a third of $\frac{7}{9}$ would be, but it doesn't fall on one of the tick marks so we cannot exactly determine where it is.

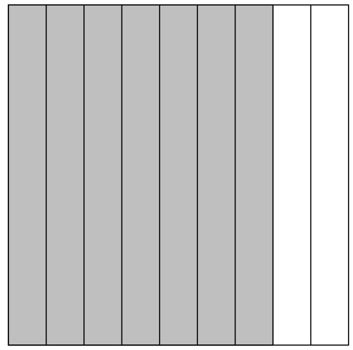


The problem here is that we can't easily break the $\frac{7}{9}$ into 3 equal pieces as we did above. We would need to alter the picture in some way to be able to find a third of it.

Question 8.6.3. How could we alter the number line above so that we can tell exactly where the blue arrow should be?

By answering the question above we see that it is possible to compute products even if they don't come out nicely. However, you may have found that your number line starts to get a little busy with all the added lines. We certainly can continue to work on the number line, but we have a second option that in the case of multiplication works nicely because it alleviates some of the clutter that arises in problems like this on the number line.

Rather than use a number line, let's go two dimensional and use a square as our whole. The shaded region below represents $\frac{7}{9}$.



To compute $\frac{1}{3} \times \frac{7}{9}$ we still want to find one third of $\frac{7}{9}$. Therefore we want to find $\frac{1}{3}$ of the shaded region.

Question 8.6.4. How does going to two dimensions make it easier for us to find a third of $\frac{7}{9}$?

Once you have used the square to find $\frac{1}{3} \times \frac{7}{9}$ think about how you would find $\frac{2}{3} \times \frac{7}{9}$ instead. Notice that the lines we need to draw are the same as before but just the amount we end up shading over changes.

Before continuing, you should take some time to compute the following products using this square model.

$$\frac{2}{5} \times \frac{3}{4} \qquad \frac{3}{4} \times \frac{8}{11} \qquad \frac{5}{7} \times \frac{8}{9} \qquad \frac{2}{3} \times \frac{7}{4}$$

The first three products should have gone exactly like the previous example. The last product however is slightly different. In order to find $\frac{2}{3}$ of $\frac{7}{4}$ we would first need to represent $\frac{7}{4}$. However, in order to do that we need two squares to start with. We have represented $\frac{7}{4}$ below.

We can now proceed to indicating $\frac{2}{3}$ of the shaded region as we did on the previous examples.

Question 8.6.5. When students attempt this problem using the above picture they sometimes think the answer is $\frac{14}{24}$. How did they come up with 24? What is the misconception that is leading to this answer?

Once we resolve the issue above, we should be able to find the product of any two fractions using this method.

Question 8.6.6. How many squares will we need to compute $\frac{4}{3} \times \frac{9}{4}$?

Let's go back to the first problem for which we used the square model, namely $\frac{1}{3} \times \frac{7}{9}$. Notice that the rectangle we end up using to find our answer is in fact a rectangle whose dimensions are $\frac{1}{3}$ by $\frac{7}{9}$. We know that to find the area of a rectangle we multiply the side lengths, so in this square model we are using we are essentially finding the area of the rectangle. For this reason this method (with some minor adjustments perhaps) is often referred to as the area model for multiplication.

Although we certainly could continue to find products using this method, there is most definitely a shortcut hidden in this method. If we look back at all of the examples we did, we can reason out what and why the numerator and denominators come out as they do. In other words we should be able to justify the following.

Property 8.6.7 (Multiplication of Fractions Property)**.** For any fractions $\frac{a}{b}$ and $\frac{c}{d}$, we have the following.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Now that we have a property that allows us to quickly compute a product, we should explore a connection between this property and the Equivalent Fractions Property. When students are learning about equivalent fractions, teachers sometimes say that the method works because they are multiplying by 1. Although this statement is true, it is not a good explanation for students who are just starting to learn about fractions. Let's explore this a bit.

Below, two different students have given their steps in showing that $\frac{2}{3}$ is equal to $\frac{8}{12}$.

Student 1	Student 2
$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$	$\frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3} \times \frac{4}{4} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$

Question 8.6.8. What are the justifications for each of the steps in the students' work above?

Question 8.6.9. Why is using the statement "We're just multiplying by 1" not a good explanation of equivalent fractions when students are just starting to learn fractions?

To really solidify this idea, it would be a good idea to be sure you recall how to justify that $\frac{2}{3} = \frac{2 \times 4}{3 \times 4}$. In other words, remind yourself how we justified the Equivalent Fractions Property *without* using multiplication of fractions.

Finally, to be clear, once students *do* understand how to multiply fractions, then it is really cool for them to see the connection back to the Equivalent Fractions Property. In fact, once they see that it is just multiplying by one, they may think about equivalent fractions in that way from then on, which is fine because it was built on the understanding of the connection, not just memorizing.

8.7 Division

In Section 8.3 we were able to compute answers to some division problems involving fractions. We now want to take the work we did back in that section and try to generalize those ideas. In so doing, we should be able to develop properties that allow us to more efficiently perform division involving fractions. Our work in that section involved computing the quotient by choosing an appropriate method of division. Recall that the two methods we used were division by grouping and division by sharing. If needed, you should return to Section 8.3 to review how we used the two methods.

Problem 10 in the Exercise Section 8.9 involved computing $5 \div 8$ because we had 5 cookies to share with 8 friends. We found that each friend got $\frac{5}{8}$ of a cookie. In other words, the answer to $5 \div 8$ ended up being $\frac{5}{8}$. Similarly, if we were to compute $3 \div 4$ we would get $\frac{3}{4}$. Therefore we see that there is a generalization that could be made here. If we carefully analyze how we computed $5 \div 8$, we should be able to generalize that method to any whole number division problem, $a \div b$, and we should be able to argue that the answer will be $\frac{a}{b}$. In other words, we have a method that allows us to justify the following property.

Property 8.7.1. [Equivalence of a division problem and a fraction] For any number a and non-zero number b, we have the following.

$$a \div b = \frac{a}{b}$$

The cookie problem allowed us to understand Property 8.7.1 using just our understanding of whole number division and the definition of a fraction. Therefore students have the knowledge to understand that a fraction is equivalent to a division problem before they ever even learn to operate with fractions. This understanding can be revisited and solidified once students understand multiplication of fractions as well. When computing $5 \div 8$, we have an amount of 5 and want to break it into 8 equal groups. In other words, we have an amount of 5 and want to find one eighth of that so that we can determine how much would be in a single group. In other words, the amount in a single group will be one eight of 5, or $5 \times \frac{1}{8}$. However, if students understand how to multiply fractions, then they know $5 \times \frac{1}{8} = \frac{5}{8}$. Therefore we have a second argument as to why $5 \div 8$ is equal to $\frac{5}{8}$.

Notice that this second argument gives us our first indication of the relationship between dividing by a number and multiplying by its reciprocal. Recall that the **reciprocal of** $\frac{a}{b}$ is the fraction $\frac{b}{a}$. In our argument above, we had an intermediate step that showed $5 \div 8$ is equal to $5 \times \frac{1}{8}$. We could generalize that idea to show that if a and b are whole numbers, then $a \div b$ is equal to $a \times \frac{1}{b}$. Let's continue to explore the connection between division and reciprocals.

Back in Section 8.3, we computed $3 \div \frac{1}{4}$. One way to compute this was to use division by grouping, so we asked ourselves how many fourths are in 3. We know that there are 4 pieces (of size one fourth) in one whole, but we have 3 wholes so we have 3×4 pieces total. Therefore we see that $3 \div \frac{1}{4}$ is equal to 3×4. We could use the same type of argument to see that $2 \div \frac{1}{5}$ will equal 2×5. Therefore we should be able to make a generalization here, but we should be clear for which type of division problems this method works. We see that the first number in the division problem is a whole number. The second number in the division problem is a fraction whose numerator is 1. A fraction whose numerator is 1 is called a **unit fraction**. Using the method described above, we should now be able to justify the following property.

Property 8.7.2. [Division of a whole number by a unit fraction] For any whole number a and fraction $\frac{1}{b}$, we have the following.

$$a \div \frac{1}{b} = a \times b$$

Question 8.7.3. The previous paragraph used division by grouping to justify Property 8.7.2. How would a justification using division by sharing go?

It is important to note that our justification of the property above relies on the fact that we have a one in the numerator, so that means we still have some work to do. We can use the property to compute $2 \div \frac{1}{4}$, so we know this equals 2×4. But what about just a tiny change to $2 \div \frac{3}{4}$. How could we compute this quotient?

Well, let's strategize a bit. We know that $\frac{3}{4}$ is just 3 pieces of $\frac{1}{4}$. So we could first find how many fourths are in 2. Once we have found that, we want to make groups of 3 from those pieces so that we have 3 fourths. This sounds like a viable strategy, so let's carefully lay out our plan.

> We will use division by grouping to compute $2 \div \frac{3}{4}$. We can determine how many $\frac{3}{4}$ are in 2 by completing the two steps below.
>
> 1. Determine how many pieces of $\frac{1}{4}$ are in 2.
> 2. Determine how many groups of 3 we can make from the number of $\frac{1}{4}$ pieces found in Step 1.

Question 8.7.4. What is the arithmetic that corresponds to Step 1 and Step 2 above?

Question 8.7.5. How does your answer to the above question show that $2 \div \frac{3}{4}$ is equal to $2 \times \frac{4}{3}$?

The strategy we used to see the reciprocal behavior relied on our use of division by grouping. We could have reached the same end goal using a different strategy if we instead used division by sharing.

Question 8.7.6. How could we see that $2 \div \frac{3}{4}$ is equal to $2 \times \frac{4}{3}$ if we were using division by sharing.

With all the work we have done above, it seems we are well on our way to a property that allows us to efficiently divide two fractions. The only issue left is that in all of our examples, the first number in the division problem was a whole number. However, if we go back and explore our explanations above, they still work even if the first number is not a whole number. For example, consider $\frac{2}{3} \div \frac{1}{4}$. Using arguments similar to above, we should still be able to argue that $\frac{2}{3} \div \frac{1}{4}$ is equal to $\frac{2}{3} \times 4$.

Question 8.7.7. In explaining why $\frac{2}{3} \div \frac{1}{4}$ is equal to $\frac{2}{3} \times 4$, do you think division by grouping or division by sharing works better?

With the discussion above then, we see that we have done enough work to finally justify the following property.

Property 8.7.8 (Division of Fractions)**.** For any fraction $\frac{a}{b}$ and non-zero fraction $\frac{c}{d}$, we have the following.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

Over the last few sections we have developed several properties that helps us to efficiently compute with fractions. It is important that teachers understand the logical development of these properties and that teachers can recognize when a property is being used. In particular, there are two main issues teachers have to deal with that are tied directly to their understanding of these properties and their logical structure. First of all, when giving an explanation to a student we want to be sure that we are only using ideas that they have learned so far. This highlights the need for teachers to know where in the mathematical flow each of these properties lie. Secondly, kids can be very creative and teachers often have to determine if the method a child is using is in fact valid. This highlights the need for teachers to be able to recognize a property when it is being used or perhaps to fill in steps that a student may not have written down. We will use the example below to give you some practice recognizing what property is being used in each step.

Example 8.7.9. Steps to compute $\frac{2}{3} \div 4$ are shown below. Justify each step.

$$
\begin{array}{rll}
 & \frac{2}{3} \div 4 & \\
 & & \text{Step 1} \\
= & (2 \div 3) \div 4 & \\
 & & \text{Step 2} \\
= & \left(2 \times \frac{1}{3}\right) \div 4 & \\
 & & \text{Step 3} \\
= & \left(2 \times \frac{1}{3}\right) \times \frac{1}{4} & \\
 & & \text{Step 4} \\
= & 2 \times \left(\frac{1}{3} \times \frac{1}{4}\right) & \\
 & & \text{Step 5} \\
= & 2 \times \frac{1}{12} & \\
 & & \text{Step 6} \\
= & \frac{2}{1} \times \frac{1}{12} & \\
 & & \text{Step 7} \\
= & \frac{2}{12} & \\
 & & \text{Step 8} \\
= & \frac{1}{6} &
\end{array}
$$

Possible Solution.

Step 1: Equivalence of a division problem and a fraction.

Step 2: First thought of 3 as $\frac{3}{1}$ which is justified by the definition of a fraction, and then used Division of Fractions Property.

Step 3: First though of 4 as $\frac{4}{1}$ which is justified by the definition of a fraction, and then used Division of Fractions Property.

Step 4: Associative property of multiplication.

Step 5: Multiplication of Fractions Property.

Step 6: Rewrote 2 as $\frac{2}{1}$.

Step 7: Multiplication of Fractions Property.

Step 8: It looks like the top and bottom was divided by 2, so this would be justified by the Equivalent Fractions Property.

□

8.8 Numerators and Denominators That Are Not Whole Numbers

Recall that we could not use the definition of a fraction to give meaning to $\frac{-2}{3}$ or $\frac{2}{-3}$. Therefore we must draw from properties we have developed to give meaning to these fractions. If we want to incorporate negatives into fractions then we should first think about all of the places we could place a negative, so that we then know all of the situations for which we need to make sense. We certainly could put a negative in front of the fraction, but the fraction is made up of a numerator and a denominator, so we could put negatives there as well. That gives us four different scenarios that we need to make sense of. In the list below, we assume a and b are natural numbers.

$$-\frac{a}{b} \qquad \frac{-a}{b} \qquad \frac{a}{-b} \qquad \frac{-a}{-b}$$

Question 8.8.1. Which one of the numbers above can we make sense of using only the definition of a negative number and the definition of a fraction?

For the other three numbers above, we now know that we cannot use the definition of fractions and the definition of negative alone to make sense of them. We must draw from properties we have developed to help us give meaning to fractions whose numerators or denominators are negative. Let's start with fractions of the form $\frac{-a}{b}$.

Question 8.8.2. How could we show that $\frac{-2}{3}$ is equal to $-\frac{2}{3}$?

You are encouraged to find more than one way to answer the question above. Once you have done so however, you should be able to use a similar argument to show that $\frac{2}{-3}$ is also equal to $-\frac{2}{3}$. Therefore we could generalize this idea to see that $\dfrac{-a}{b}$, $\dfrac{a}{-b}$, and $-\dfrac{a}{b}$ are all equal.

Question 8.8.3. What is $\dfrac{-a}{-b}$ equal to?

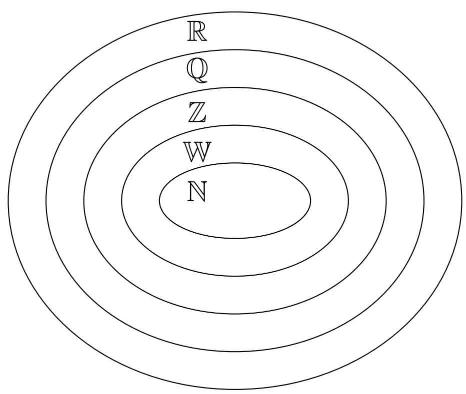
We have now made sense of fractions whose numerators or denominators may be negative, which we said we would do back when we defined what a rational number was. Let's revisit that definition. Is -3 a rational number? Well, we need to decide if -3 can be written in the form $\frac{a}{b}$ where a and b are integers. To do that, we can use the equivalence of a division problem and a fraction, so we know $\frac{-3}{1}$ is equal to $-3 \div 1$ and that of course is equal to -3. Thus -3 is equal to $\frac{-3}{1}$, and since

-3 and 1 are integers we have expressed -3 in rational form. Therefore -3 is a rational number. In this way we can see that all integers are in fact rational numbers as well.

This is a good time to pause and think about the extensions of numbers we have developed over the past two semesters. We started with the counting numbers, which we called the natural numbers. This set is often denoted by $\mathbb{N}$, so $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$. When we then included 0 into the set of numbers, we called that the set of whole numbers, denoted $\mathbb{W} = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$. Next we introduced the negatives of each of these numbers and we called that the set of integers. The standard notation for the integers uses the letter "Z" which comes from the german word Zahlen, so $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. And now, we have introduced the set of rational numbers, which is denoted by $\mathbb{Q}$. Recall that $\mathbb{Q}$ is the set of all numbers that can be expressed in the form $\frac{a}{b}$ where a and b are integers.

You may recall hearing that the number π is not a rational number. In other words, it cannot be written as a fraction whose numerator and denominator are integers. Numbers that cannot be written in rational form (i.e. integer over integer) are called **irrational numbers**. The set of rational numbers together with the set of irrational numbers is called the set of real numbers and is denoted by $\mathbb{R}$.

Therefore we have gradually been extending our number system to include more and more numbers. To be clear, by extension we mean for example that all natural numbers are whole numbers, integers, rational numbers, and real numbers. All whole numbers are integers, rational numbers, and real numbers. Etc. This extension of numbers is expressed by the diagram below.



Because there are irrational numbers we need to be careful about using the word "fraction" to mean a rational number. For example, consider the number π, which is an irrational number. However, using the fact that $\frac{a}{b}$ is equal to $a \div b$, then π is equal to $\frac{\pi}{1}$ for example. Similarly, π is equal to $\frac{2\pi}{2}$. Therefore π can be expressed in fraction form. However, the numerator is not an integer, so it is not a fraction in rational form. Most of the time when we say fraction we mean a rational fraction (i.e. an integer over an integer). However, if we are in a situation where that distinction is important we want to make sure we are clear about our use of the word fraction.

Using the equivalence of a division problem and a fraction, we know that for any non-zero real number, c, that $\frac{c}{c}$ is equal to 1. This allows us to extend our Equivalent Fractions property to allow any multiplier, not just a natural number multiplier as previously stated. The steps to show that $\frac{a}{b}$ is equal to $\frac{a \times c}{b \times c}$ are shown below.

$$
\begin{aligned}
& \frac{a \times c}{b \times c} && \text{Step 1: Multiplication of fractions (in reverse).} \\
= \quad & \frac{a}{b} \times \frac{c}{c} && \text{Step 2: Justified above.} \\
= \quad & \frac{a}{b} \times 1 && \text{Step 3: Anything multiplied by 1 is itself.} \\
= \quad & \frac{a}{b}
\end{aligned}
$$

Therefore we have justified a new statement of the Equivalent Fractions Property that allows any real number as a multiplier. This revised property is stated below.

Property 8.8.4 (Equivalent Fractions)**.** For any non-zero real number c and real number $\frac{a}{b}$, we have the following.

$$\frac{a}{b} = \frac{a \times c}{b \times c} \qquad\qquad \frac{a}{b} = \frac{a \div c}{b \div c}$$

We can make similar adjustments to all of the properties we have developed previously, so instead of a, b, c, and d needing to be whole numbers we can say that they are real numbers. This gives us the ability to work with some crazy fractions.

Question 8.8.5. Is $\frac{\frac{2}{3}}{\frac{4}{5}}$ a rational number?

Question 8.8.6. Is $\frac{\frac{2}{3}}{4}$ equal to $\frac{2}{\frac{3}{4}}$?

Question 8.8.7. Is $\frac{2\pi}{3\pi}$ a rational number?

8.9 Exercises (Part I)

When answering the questions in this section you may **not** use any of the properties stated in Section 8.4 or later.

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1. For a positive integer a, determine whether the following statements are always true, sometimes true, or never true.
   (a) $\frac{7}{a}$ is greater than $\frac{5}{a}$.
   (b) $\frac{a}{7}$ is greater than $\frac{a}{5}$.

2. Use fraction strips to answer each of the questions below.
   (a) You are given a strip of paper that you have been told has a length of $\frac{1}{6}$. Explain how you could verify that the length is in fact $\frac{1}{6}$.
   (b) You are given a strip of paper that you have been told has a length of $\frac{3}{4}$. Explain how you could verify that the length is in fact $\frac{3}{4}$.
   (c) Explain how to show that $\frac{4}{12} = \frac{1}{3}$.
   (d) Explain how to determine what whole number $\frac{9}{7}$ is closest to, and how far away from $\frac{9}{7}$ that whole number is.
   (e) You are given a blue strip of paper and a red strip of paper. The blue strip has length $\frac{1}{4}$ and the red strip has length $\frac{2}{3}$. How much longer is the red strip than the blue strip?

3. Max is comparing the fractions $\frac{5}{6}$ and $\frac{5}{8}$. He draws the following pictures and makes the conclusion that the two fractions are equal.

   (a) According to his pictures and his conclusion, what can you determine Max understands about fractions? Be sure to make it clear how your response is supported by his picture.
   (b) How could Max redraw his picture so that he comes to the correct conclusion about comparing $\frac{5}{6}$ and $\frac{5}{8}$? (In other words, what needs to be “fixed” in Max’s picture?)

4. When answering the questions below, be sure to support your answer with an explanation. You are welcome to draw pictures to help you answer any of the following questions, however be sure that your conclusion does not rely on the accuracy of your drawing.

(a) Which is larger $\frac{1}{2}$ or $\frac{45}{89}$?

(b) Which is larger $\frac{4}{9}$ or $\frac{7}{13}$?

(c) Which is larger $\frac{4}{7}$ or $\frac{6}{11}$?

(d) Which is larger $\frac{7}{9}$ or $\frac{9}{11}$?

(e) What does $\frac{1}{2} \div \frac{1}{8}$ equal?

(f) What does $\frac{2}{3} \div 4$ equal?

5. Use the definition of a fraction on the number line to answer the following questions.

   (a) Show that $\frac{2}{3}$ is equal to $\frac{8}{12}$. (This should not rely on the accuracy of your drawing. It should be clear without doubt that the two fractions are equal.)

   (b) For positive integers a, b, and n is the following statement always true, sometimes true, or never true? Justify your response.

$$\frac{a}{b} = \frac{a \times n}{b \times n}$$

   (c) For a positive integer n and positive integers a and b that are divisible by n, is the following statement always true, sometimes true, or never true? Justify your response.

$$\frac{a}{b} = \frac{a \div n}{b \div n}$$

6. (a) Use the definition of a fraction to explain why $\frac{5}{0}$ is undefined.

   (b) Use the definition of a fraction to explain why $\frac{0}{5}$ equals 0.

7. For positive integers a and b determine if the following statements are always true, sometimes true, or never true. Justify your response.

   (a) If $\frac{a}{b}$ is greater than 1, then $a > b$.

   (b) The fraction $\frac{a+3}{a+1}$ is bigger than 1.

   (c) The fraction $\frac{2a+1}{25}$ is greater than 1.

8. When answering the questions below be sure to make it clear how you arrived at your answer.

   (a) Two thirds of a square has been colored blue and the rest of the square has been colored red. Three fourths of the blue section has polka dots. What fraction of the square is blue with polka dots?

   (b) A blue segment on the number line has length $\frac{2}{3}$. How many blue segments can be made from a length of $\frac{3}{4}$? (Portions of blue segments are allowed.)

   (c) A length of $\frac{3}{4}$ makes $\frac{2}{3}$ of a red segment. How long is a red segment?

9. It takes $\frac{2}{3}$ of a pound of chocolate to make a certain type of candy bar called a JumboBar.

   (a) How many pounds of chocolate are in $\frac{3}{4}$ of a JumboBar?

   (b) If you have $\frac{3}{4}$ of a pound of chocolate, how many JumboBars could you make?

10. There are 5 identical candy bars and 8 kids who will share the candy fairly. Without doing any arithmetic, describe how you could cut the candy bars to share them. How much will each kid get?

11. Use a number line to compute the following. Be sure to make it clear how you arrived at your answer.

(a) $\frac{4}{5} + \frac{2}{3}$ (b) $\frac{4}{5} - \frac{2}{3}$ (c) $3\frac{1}{2} - 2\frac{2}{3}$

12. Compute the following. Be sure to explain how you arrived at your answer.

(a) $\frac{12}{5} \div \frac{3}{5}$ (c) $\frac{12}{5} \div 4$ (e) $2\frac{1}{3} \times \frac{3}{5}$ (g) $\frac{2}{5} \div \frac{2}{3}$
(b) $3 \times \frac{2}{5}$ (d) $\frac{2}{3} \times \frac{4}{5}$ (f) $2\frac{5}{8} \div \frac{3}{4}$ (h) $\frac{2}{5} \div \frac{4}{7}$

13. Please make it clear how you arrived at each of your answers below.

(a) Compute $3 \times \frac{2}{5}$.
(b) Compute $\frac{1}{3} \times \frac{6}{11}$.
(c) Without actually finding the answer to $\frac{2}{3} \div \frac{4}{5}$, determine which is bigger $\frac{2}{3} \div \frac{4}{5}$ or 1?
(d) Without actually finding the answer to $5 \div \frac{2}{3}$, determine which is bigger $5 \div \frac{2}{3}$ or 5?

14. (a) Without drawing a picture explain how you know that $\frac{27}{5}$ is equal to $5\frac{2}{5}$.

(b) According to text books, to convert a mixed number to an improper fraction you are supposed to multiply the whole number by the denominator and add the numerator. This gives you your new numerator for the improper fraction. The denominator stays the same.

So for example to convert $7\frac{2}{5}$ to an improper fraction you would compute $7 \times 5 + 2 = 37$. So the improper fraction would be $\frac{37}{5}$.

Explain why the whole number times the denominator plus the numerator is the new numerator, and explain why the denominator stays the same.

(c) According to text books, to convert an improper fraction to a mixed number you divide the numerator by the denominator. The quotient becomes your whole number and the remainder becomes your new numerator. The denominator stays the same.

For example, to convert $\frac{44}{9}$ to a mixed number you would compute $44 \div 9 = 4$ r.8. So the mixed number is $4\frac{8}{9}$.

Explain why the quotient is the whole number, why the remainder is the numerator, and why the denominator stays the same.

15. (a) According to text books, if you multiply the numerator and denominator by the same number you get an equal fraction. So for example $\frac{3}{5} = \frac{3 \cdot 28}{5 \cdot 28}$. Use a number line argument to show that $\frac{3}{5}$ is in fact equal to $\frac{3 \cdot 28}{5 \cdot 28}$. Please do not actually draw the number line just explain what would be done with it to see this equality.

    (b) According to text books, if you divide the numerator and denominator by the same number you get an equal fraction. So for example $\frac{45}{55} = \frac{45 \div 5}{55 \div 5}$. Use a number line argument to show that $\frac{45}{55}$ is in fact equal to $\frac{45 \div 5}{55 \div 5}$. Please do not actually draw the number line just explain what would be done with it to see this equality.

16. For positive integers a and b with $a < b$, is the following statement always true, sometimes true, or never true. Justify your response.

    The fraction $\frac{a}{b}$ is closer to 1 than the fraction $\frac{b}{a}$.

17. For a positive integer a, explain why the following statements will always be true.

    (a) $5 \div \frac{1}{a}$ is equal to $5 \times a$.

    (b) $3 \div a$ is equal to $\frac{3}{a}$.

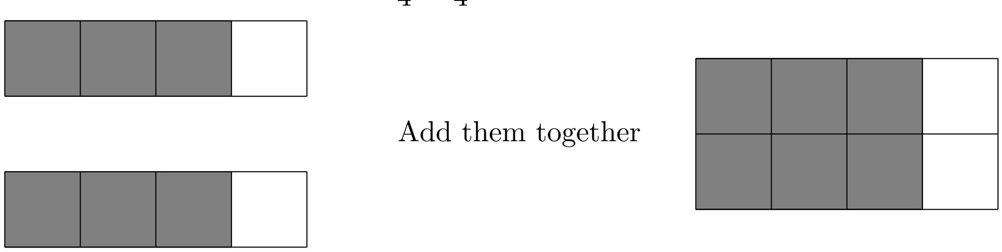
18. (a) Use the definition of a fraction to place $\dfrac{3\frac{1}{4}}{5}$ on the number line. Be sure that your picture clearly shows this specifically named fraction.

    (b) Find a fraction in rational form that is equal to $\dfrac{3\frac{1}{4}}{5}$. Please be sure to show enough work so that it is clear how you arrived at your answer.

    (c) Use the definition of a fraction to place $\dfrac{3}{2\frac{1}{2}}$ on the number line. Be sure that your picture clearly shows this specifically named fraction.

    (d) Find a fraction in rational form that is equal to $\dfrac{3}{2\frac{1}{2}}$. Please be sure to show enough work so that it is clear how you arrived at your answer.

19. Max is working on computing $\dfrac{3}{4} + \dfrac{3}{4}$. He draws the following picture.



He then says, "All together I end up with 6 pieces out of 8, so $\frac{3}{4} + \frac{3}{4}$ is equal to $\frac{6}{8}$."

Think of what you could ask Max that would help him to see his mistake. Please record this as a back and forth conversation that allows Max to recognize his error and correct it, without being told.

8.10 Exercises (Part II)

When answering the questions in this section you may use any of the properties stated in Chapters 1.1 - 8, unless otherwise noted.

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1. Max and Macy are talking about how to decide which of two fractions is bigger.

   (a) Is Max's statement below always true, sometimes true, or never true? Justify your response.

   The fraction with the bigger numerator is the bigger fraction.

   (b) Is Macy's statement below always true, sometimes true, or never true? Justify your response.

   The fraction with the smaller denominator is the bigger fraction.

   (c) After much discussion, Max and Macy together come up with the statement below. Is the statement always true, sometimes true, or never true? Justify your response.

   Just look at the "missing pieces". The fraction with fewer missing pieces is the bigger fraction.

   (For example, consider $\frac{4}{7}$ and $\frac{8}{9}$. In $\frac{4}{7}$ there are 3 missing pieces to make a whole. In $\frac{8}{9}$ there is only 1 missing piece. Therefore $\frac{8}{9}$ has fewer missing pieces than $\frac{4}{7}$, so according to the statement above $\frac{8}{9}$ is the bigger fraction.)

2. Use a square as your whole to answer the following questions. For this problem, you may not use the Multiplication of Fractions Property (Property 8.6).

   (a) What does $\frac{2}{3} \times \frac{4}{5}$ mean and how can it be represented on the square?

   (b) The answer to this multiplication problem is $\frac{8}{15}$. Explain, using your picture, why the numerator is 8 and why the denominator is 15.

   (c) Use your picture to explain why the 8 came from 2×4.

   (d) Use your picture to explain why the 15 came from 3×5.

3. For this problem, you may not use the Multiplication of Fractions Property (Property 8.6).

   (a) Using a square as your whole, represent the multiplication problem $2\frac{2}{3} \times 3\frac{4}{5}$?

(b) What is the answer to this multiplication problem and how can you get that answer from your picture drawn above?

(c) A student computed this product by computing 2×3 and $\frac{2}{3} \times \frac{4}{5}$, so he thinks the answer is $6\frac{8}{15}$. Indicate on your picture what portions of the answer this student found.

(d) According to your picture, what products does this student still need to compute in order to complete his answer?

4. Determine which fraction is larger without using a common denominator, cross multiplying, or converting to decimals. Explain how you arrived at your answer.

(a) $\frac{4}{11}$ or $\frac{1}{3}$

(b) $\frac{5}{16}$ or $\frac{1}{3}$

5. For this problem you may not use the Division of Fractions Property (Property 8.7). Be sure to make it clear how you arrived at your answer.

(a) Use division by grouping to compute $\frac{4}{5} \div \frac{1}{10}$.

(b) Use division by sharing to compute $\frac{4}{5} \div \frac{1}{10}$.

(c) Which of the two methods above do you think provides a justification for why $\frac{4}{5} \div \frac{1}{10}$ is equal to $\frac{4}{5} \times 10$? Use that method to justify why $\frac{5}{7} \div \frac{1}{4}$ is equal to $\frac{5}{7} \times 4$.

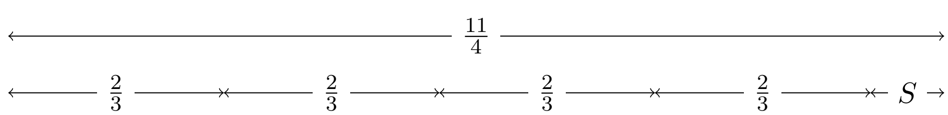
6. For this problem you may not use the Division of Fractions Property (Property 8.7). Be sure to make it clear how you arrived at your answer.

(a) Use division by grouping to compute $\frac{4}{5} \div \frac{2}{3}$.

(b) Use division by sharing to compute $\frac{4}{5} \div \frac{2}{3}$.

(c) Which of the two methods above do you think provides a justification for why $\frac{4}{5} \div \frac{2}{3}$ is equal to $\frac{4}{5} \times \frac{3}{2}$? Use that method to justify why $\frac{5}{7} \div \frac{3}{4}$ is equal to $\frac{5}{7} \times \frac{4}{3}$.

7. (a) In the drawing below, what does S equal?



(b) Computing $\frac{11}{4} \div \frac{2}{3}$, we get $\frac{11}{4} \div \frac{2}{3} = \frac{11}{4} \times \frac{3}{2} = \frac{33}{8} = 4\frac{1}{8}$. How are S and $\frac{1}{8}$ related?

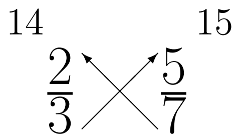
8. Imagine that you are teaching division of fractions. To make this meaningful for kids, many teachers try to relate mathematics to other things. Sometimes they try to come up with real-world situations or story problems to show the application of some particular piece of content.

(a) What would you say would be a good story problem for $1\frac{2}{3} \div \frac{1}{2}$?

(b) Is your story problem an example of division by sharing or division by grouping?

(c) One way to write the answer to this division problem is $3\frac{1}{3}$. In the context of your story problem, what is the meaning of the fraction $\frac{1}{3}$?

9. Max learned this cool "cross multiply" trick to determine which fraction is larger. Here's what he does:



Since 14 is smaller than 15, then $\frac{2}{3}$ is smaller than $\frac{5}{7}$.

Explain why the "cross multiply" trick works. Be sure to explain why it will **always** work, not just in this example.

10. (a) Find three numbers between $\frac{7}{9}$ and $\frac{8}{9}$. Show all work, and please state your answers in rational form.

(b) Find three numbers (different from those you found above) between $\frac{7}{9}$ and $\frac{8}{9}$. Show all work, and please state your answers in rational form.

(c) Describe a method that would allow you to find any amount of numbers between $\frac{7}{9}$ and $\frac{8}{9}$. Be sure to justify why your method works.
(For example, if I asked you to find 23 numbers between, you could apply your method to do so. Or if I asked for 51 numbers between, your method would work for that as well.)

(d) Does the method you described above result in numbers that are evenly spaced apart? If so, describe an alternate method that would result in numbers that are not evenly spaced. If not, describe an alternate method that would result in numbers that are evenly spaced. Be sure to justify why your alternate method works.

11. You give Max the problem $2\dfrac{3}{4} + 1\dfrac{1}{5}$. Max does the problem in the following way.

$$2 + 1 = 3$$
$$\frac{3}{4} + \frac{1}{5} = \frac{15}{20} + \frac{4}{20} = \frac{19}{20}$$
Therefore Max concludes that $2\dfrac{3}{4} + 1\dfrac{1}{5} = 3\dfrac{19}{20}$.

(a) Is Max's method and answer correct? If yes, why does it work? If not, explain why not.

(b) You then give Max the problem $2\dfrac{3}{4} + 1\dfrac{2}{5}$. He tries to do it the same way, but gets confused. What could be confusing Max? What can you say to help him?

12. You give Max the problem $-2\dfrac{3}{4} + 1\dfrac{1}{5}$. Max does the problem in the following way.

$$-2 + 1 = -1$$

$$\frac{3}{4} + \frac{1}{5} = \frac{15}{20} + \frac{4}{20} = \frac{19}{20}$$

Therefore Max concludes that $-2\frac{3}{4} + 1\frac{1}{5} = -1\frac{19}{20}$.

Is Max's method and answer correct? If yes, why does it work? If not, explain why not and give the correct answer.

13. You give Max the problem $2\frac{3}{4} \times 1\frac{1}{5}$. Max does the problem in the following way.

$$2 \times 1 = 2$$

$$\frac{3}{4} \times \frac{1}{5} = \frac{3}{15}$$

Therefore Max concludes that $2\frac{3}{4} \times 1\frac{1}{5} = 2\frac{3}{15}$.

Is Max's method and answer correct? If yes, why does it work? If not, explain why not and give the correct answer.

14. Consider the following story problem.

> Kevin has $3\frac{2}{5}$ pounds of candy corn. He gives $\frac{1}{4}$ of his candy to his brother, Tyler. How many pounds of candy does Kevin have left?

(a) In the context of the problem, what does $\frac{1}{4} \times 3\frac{2}{5}$ represent?

(b) Max and Macy's work to solve the story problem is shown below. They used different strategies but both arrived at the same answer. Describe in your own words each of their methods.

Macy's work

$$\frac{1}{4} \times 3\frac{2}{5} = \frac{1}{4} \times \frac{17}{5} = \frac{17}{20}$$

$$3\frac{2}{5} - \frac{17}{20} = 3\frac{8}{20} - \frac{17}{20} = 2\frac{11}{20}$$

Kevin has $2\frac{11}{20}$ pounds of candy left.

Max's work

$$\frac{3}{4} \times 3\frac{2}{5} = \frac{3}{4} \times \frac{17}{5} = \frac{51}{20} = 2\frac{11}{20}$$

Kevin has $2\frac{11}{20}$ pounds of candy left.

(c) If Macy used her same method as above, what would her solution to the following story problem look like?

> Timmy has $3\frac{1}{2}$ pounds of sand piled up for his sand castle. His mom makes him give $\frac{2}{5}$ of his pile to his sister. How many pounds of sand does Timmy have left?

(d) If Max used his same method as above, what would his solution to the above story problem look like?

(e) Look back at the original candy corn problem. Rewrite that story problem, **changing as few words as possible**, so that the answer to the new story problem will be found by computing $3\frac{2}{5} - \frac{1}{4}$.

15. Consider the following start to a story problem.

> Max has $2\frac{3}{5}$ pounds of dog food. A daily serving of dog food for his pet St. Bernard is $\frac{2}{3}$ of a pound.

(a) What question could you complete this story problem with in order to make it a word problem whose solution is $2\frac{3}{5} - \frac{2}{3}$?

(b) What question could you complete this story problem with in order to make it a word problem whose solution is $2\frac{3}{5} \div \frac{2}{3}$?

(c) The answer to $2\frac{3}{5} \div \frac{2}{3}$ is $3\frac{9}{10}$. Provide two questions to complete this story problem so that the answer to the first question is 3 and the answer to the second question is $\frac{9}{10}$.

(d) In the context of the original scenario and your questions in part 15c, what does $\frac{9}{10} \times \frac{2}{3}$ represent?

16. Consider the following story problem.

> A machine is making ribbons for bows when the machine breaks down. At that moment the ribbon is $\frac{2}{5}$ of yard long, and the readout on the screen says the job is $\frac{5}{8}$ complete. How long is a completed ribbon?

(a) A fellow student says that the solution to this problem can be found by computing $\frac{2}{5} \div \frac{5}{8}$. If that is true, then this story problem must either fit into the division by sharing category or the division by grouping category. Into which category does this problem fall? Explain why it fits into that category.

(b) At one point this machine only has enough thread left to create 3 yards of ribbon. How many completed ribbons will the machine be able to make?

17. Max knows that $\frac{2}{3} = \frac{10}{15}$ because we multiplied the top and bottom by 5. He also knows that $\frac{24}{30} = \frac{4}{5}$ because we divided the top and bottom by 6. After seeing this Max says, "Oh...As long as we do the same thing to the top and the bottom, we get equal fractions." Let's explore this idea by determining whether the following statements are always true, sometimes true, or never true. Justify your response.

    (a) If we add the same nonzero number to the top and the bottom of a fraction, then we will get a fraction equal to the original fraction.

    (b) If we square the top and the bottom of a nonzero fraction, then we will get a fraction equal to the original fraction.

18. (a) Show that $\left(-\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$. Be sure to justify each step you make.

    (b) Show that $\left(-\frac{a}{b}\right)^5 = -\frac{a^5}{b^5}$. Be sure to justify each step you make.

19. Consider the problem $\frac{8}{15} \times \frac{9}{14}$.

    (a) Using properties developed in the book, give a step by step computation of this problem where your final answer is a fraction in simplest form. Justify each step.

    (b) Max has learned a shortcut method to get to the answer to this product by canceling things out first. He calls this method the "cross-cancel" method, and his work is shown below.

    (c) Given the product of any two fractions, explain why the answer found using the cross-cancel method will always produce the same answer as the standard method shown in part 19a.

    (d) Is the following statement always true, sometimes true, or never true? Justify your response.

    When computing the sum of two fractions, the cross-cancel method will produce the same answer as the standard method.

20. Below is an alternate proof of the equivalence of a division problem and a fraction (Property 8.7.1). Each step in the proof is labeled. Give a justification for each step.

$$\underbrace{a \div b = a \div \frac{b}{1}}_{\text{step 1}} \overbrace{}^{} = \underbrace{a \times \frac{1}{b} = \frac{a}{1} \times \frac{1}{b}}_{\text{step 3}} = \underbrace{\frac{a \times 1}{1 \times b} = \frac{a}{b}}_{\text{step 5}}$$

$$\text{step 2: } a \div \frac{b}{1} = a \times \frac{1}{b}, \quad \text{step 4: } \frac{a}{1} \times \frac{1}{b} = \frac{a \times 1}{1 \times b}$$

21. In the above problem we showed $a \div b$ is equal to $\frac{a}{b}$ by starting with $a \div b$ and taking steps to turn that expression into $\frac{a}{b}$.

(a) Give a step by step proof (as in Problem 20) to show that $\frac{a}{c} \div \frac{b}{c}$ is equal to $a \div b$. Be sure to justify each step.

(b) Is it true that $\frac{a}{c} \times \frac{b}{c}$ is equal to $a \times b$? If so, prove it. If not, explain why we could ignore the denominator when dividing the fractions (problem 21a) but could not ignore the denominator when multiplying the fractions.

22. (a) Justify each step in the computation below.

$$\frac{6}{8} = \frac{3 \times 2}{4 \times 2} = \frac{3}{4} \times \frac{2}{2} = \frac{3}{4} \times 1 = \frac{3}{4}$$

(b) Students learn the Equivalent Fractions Property in fourth grade. Students do not learn the Multiplication of Fractions Property until fifth grade. Explain why the method above to show that $\frac{6}{8} = \frac{3}{4}$ is not a valid fourth grade argument.

23. (a) Explain in your own words what it means to say that one integer is "next" to another.

(b) Can this definition of "next" be extended to rational numbers? For example, is there a rational number next to $\frac{2}{5}$?

9 Exponents (Part II)

9.1 Extending the Basics

When we were first introduced exponents back in Chapter 5 we had not yet dealt with fractions. Now that we understand fractions, let's revisit the basics of exponents in light of our new found understanding of fractions. Before proceeding, you are encouraged to review the definition and properties developed in Chapter 5.

Question 9.1.1. Suppose $\frac{a}{b}$ is some rational number and n is some natural number. Is the following statement always true, sometimes true, or never true?

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

There are many problems like the one above, where we just need to take our understanding of natural number exponents and our understanding of fractions and combine them to reason out some new ideas.

Back in Chapter 5 we developed Property 5.2.9 which showed the relationship between division and exponents. However, now that we know fractions are equivalent to division problems, we should revisit this property in light of fractions. The property stated that $a^n \div a^m = a^{n-m}$, and we used the definition of division to reason this out. Let's now reconsider this property using our knowledge of fractions.

Let's consider the example $2^5 \div 2^3$. According to Property 5.2.9, $2^5 \div 2^3$ is equal to 2^{5-3}. However, using the equivalence of a division problem and a fraction $2^5 \div 2^3$ is equal to $\dfrac{2^5}{2^3}$. Therefore $\dfrac{2^5}{2^3}$ must be equal to 2^{5-3}. Let's make sure this coincides with our understanding of fractions. The fraction $\dfrac{2^5}{2^3}$ can be written out as $\dfrac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2}$, and then using the Equivalent Fractions Property, we can cancel the three 2's on the bottom with three of the 2's on top. In other words we have five 2's on top and we are going to take away three 2's. Thus we do in fact see the $5 - 3$ showing up when we think about it as fractions as well. Therefore we can rewrite Proposition 5.2.9 in terms of fractions, and we will call this the Subtractive Law of Exponents.

Property 9.1.2 (Subtractive Law of Exponents)**.** For any numbers a^n and a^m, we have the following.

$$\frac{a^n}{a^m} = a^{n-m}$$

Interestingly enough, the Subtractive Law of Exponents gives us a way to understand negative exponents. Recall that the definition of exponents given back in Chapter 5 did not apply to an exponent of 0, or negative exponents, or fractional exponents. Back in that chapter, we were able to reason out what a^0 must be equal to, and you are encouraged to review that process before moving forward.

Question 9.1.3. How can we use the Subtractive Law of Exponents to reason out what 2^{-3} must equal?

Question 9.1.4. How can we use the Subtractive Law of Exponents to reason out what $\left(\frac{2}{5}\right)^{-3}$ must equal?

Depending on the method you chose, you may have found that Question 9.1.3 was easier than Question 9.1.4. However, after careful analysis of Question 9.1.4, we can generalize those ideas to give meaning to a negative exponent. We officially state the property for negative exponents below.

Property 9.1.5 (Negative Exponents)**.** For any number $\frac{a}{b}$ and positive integer n, we have the following.

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

In the statement above notice that our base is a fraction. What if our base is a whole number? In fact, sometimes textbooks make two different "rules" for negative exponents, one for a fractional base and one for a whole number base. However, that is completely unnecessary.

Question 9.1.6. Suppose a student has memorized Property 9.1.5, and that is the only thing she remembers about negative exponents. So for example, when she sees 5^{-4}, she thinks she is stuck. How could that student apply Property 9.1.5 to show that $a^{-n} = \dfrac{1}{a^n}$?

Comparing Question 9.1.3 and Question 9.1.6, we see that Question 9.1.3 is one way a student can reason out what a negative exponent means when they are first exposed to negative exponents. However, a few years down the road, many students just have the rule that "negative exponents flip the fraction" memorized. Sometimes those students get stumped by a whole number to a negative exponent and so Question 9.1.6 allows them to get unstuck using the rule they remember.

9.2 Fractional Exponents

Just as with negative exponents, the definition of exponents does not apply to fractional exponents. Therefore we need to draw from properties we have developed to reason out what fractional exponents must mean. In particular, we will use the Additive Law of Exponents, so let's first do a few examples.

Example 9.2.1. According to the Additive Law of Exponents, what does $5^{\frac{1}{2}} \times 5^{\frac{2}{3}}$ equal in exponent form?

Answer. $5^{\frac{1}{2}} \times 5^{\frac{2}{3}} = 5^{\frac{1}{2}+\frac{2}{3}} = 5^{\frac{3}{6}+\frac{4}{6}} = 5^{\frac{7}{6}}$.

Example 9.2.2. According to the Additive Law of Exponents, what does $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}}$ equal in exponent form?

Answer. $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 8^1$

Question 9.2.3. How does the last example help you determine what $8^{\frac{1}{3}}$ must equal?

Question 9.2.4. How could you use a similar idea to determine what $32^{\frac{1}{5}}$ must equal?

Question 9.2.5. When using the same ideas as above to determine $9^{\frac{1}{2}}$, there is a subtle issue that arises. What is it?

At this point we need to introduce some vocabulary and some notation. Imagine we are looking for a number, that when cubed, is equal to 27. (The number is 3, but that's not the important thing right now.) We call that number the cube root of 27, and the notation we use is $\sqrt[3]{27}$. In other words $\sqrt[3]{27}$ is the solution to the equation $x^3 = 27$. Notice that there is exactly one solution to that equation, namely $x = 3$.

On the other hand, consider the following scenario. Suppose we are looking for a number, that when squared, is equal to 25. In other words, we are looking for a solution to the equation $x^2 = 25$. How many solutions does this equation have? At first glance you might think $x = 5$, but then you realize that $(-5)^2$ is always equal to 25. Thus $x^2 = 25$ has two solutions. We use the term square root to indicate the positive solution to that equation. The notation we use is $\sqrt{25}$. So when we write $\sqrt{25}$ we mean the positive solution to the equation $x^2 = 25$.

Question 9.2.6. Why did we have an issue like this with square root but not with cube root?

We now give the formal definition of roots for any natural number 2 or bigger.

Definition 9.2.7 (The nth Root)**.** For any natural number $n \geq 2$, we define the nth root of a, denoted $\sqrt[n]{a}$, as follows.

- If n is odd, then $\sqrt[n]{a}$ is the solution to the equation $x^n = a$.
- If n is even, then $\sqrt[n]{a}$ is the non-negative solution to the equation $x^n = a$.

When $n = 2$ we call it the square root rather than the second root and write $\sqrt{a}$ rather than $\sqrt[2]{a}$. When $n = 3$ we often call it the cube root instead of the third root. One note about the language used in the second bullet. Notice that we said non-negative rather than positive. We use this language so that the second bullet point covers the case when $a = 0$. In particular $\sqrt[n]{0}$ means the solution to $x^n = 0$, which always only has one solution, namely $x = 0$, which is not positive, but is non-negative. Thus $\sqrt[n]{0} = 0$.

With these definitions in hand we can now state the property that describes the behavior of fractional exponents. Be sure in answering Questions 9.2.3 - 9.2.5, you see how to justify the property below.

Property 9.2.8 (Fractional Exponents)**.** For any natural number $n \geq 2$ and any number a, we have the following.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Question 9.2.9. How can we use exponent properties developed in Chapter 5 and Property 9.2.8 to determine what $27^{\frac{2}{3}}$ equals?

Depending on your method of choice in answering the question above, your work will support the fact that $27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2$ or that $27^{\frac{2}{3}} = \sqrt[3]{27^2}$. In general, as long as a is a positive number then

$\sqrt[m]{a^n}$ is equal to $(\sqrt[m]{a})^n$. Unfortunately, since we are dealing with square or higher roots, negative numbers can cause some problems.

Consider the example $(-9)^{\frac{2}{4}}$. Since $\frac{2}{4} = \frac{1}{2}$, then $(-9)^{\frac{2}{4}} = (-9)^{\frac{1}{2}} = \sqrt{-9}$ which is undefined. Therefore $(-9)^{\frac{2}{4}}$ is undefined. This matches our conclusion if we use the definition $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$ because $(-9)^{\frac{2}{4}} = \left(\sqrt[4]{-9}\right)^2$ and $\sqrt[4]{-9}$ is undefined. However, if we use the definition $a^{\frac{n}{m}} = \sqrt[m]{a^n}$, we get the following: $(-9)^{\frac{2}{4}} = \sqrt[4]{(-9)^2} = \sqrt[4]{81} = 9$. So in this case we got a different result for $(-9)^{\frac{2}{4}}$...that's bad!

To summarize, we can say that if we are working with a positive base then $a^{\frac{n}{m}}$ can be defined as either $\sqrt[m]{a^n}$ or $(\sqrt[m]{a})^n$, and so you should use whichever one suits the problem best. (In other words, you have flexibility here!) However, if we are working with a negative base, then we will need to define $a^{\frac{n}{m}}$ as $(\sqrt[m]{a})^n$.

9.3 Exercises

1. Determine whether the following statements are always true, sometimes true, or never true.

   (a) The statement given in Question 9.1.1

   (b) For nonzero real numbers a and b, $\dfrac{a^n}{b^m} = \left(\dfrac{a}{b}\right)^{n-m}$.

2. Answer the given questions found in the text. Be sure to show and justify every arithmetic step you make.

   (a) Question 9.1.3. (b) Question 9.1.4. (c) Question 9.1.6.

3. In Problem 2 we were able to reason out the meaning of a negative exponent. The sequence of problems below provide an alternate method to determine what negative exponents mean.

   (a) According to the Additive Law of Exponents, what does $\left(\frac{2}{3}\right)^{-1} \times \left(\frac{2}{3}\right)^{1}$ equal?

   (b) How can you use your answer to the question above to determine what $\left(\frac{2}{3}\right)^{-1}$ must equal?

4. Answer the given questions found in the text.

   (a) Question 9.2.3. (b) Question 9.2.4. (c) Question 9.2.5.

5. (a) Answer Question 9.2.9 in the text.

   (b) What definition for $a^{\frac{n}{m}}$ is supported by your work above?

   (c) Give an alternate answer to Question 9.2.9 so that your work supports the other definition for $a^{\frac{n}{m}}$.

6. Answer the following questions without use of a calculator.

   (a) What two whole numbers does $\sqrt{15}$ lie between? Show enough work so that I can tell how you arrived at your answer.

   (b) Explain why $\sqrt{-15}$ is undefined.

   (c) Is $\sqrt[5]{-15}$ undefined? Why or why not?

7. For a real number x determine whether the following statements are always true, sometimes true, or never true. Justify your response.

   (a) As long as $\sqrt{x^2}$ is defined, then $\sqrt{x^2} = x$

   (b) As long as $\sqrt[3]{x^3}$ is defined, then $\sqrt[3]{x^3} = x$

   (c) As long as $\sqrt{x}$ is defined, then $(\sqrt{x})^2 = x$.

8. For positive real numbers x and y, are the following statement always true, sometimes true, or never true? Justify your response.

   (a) $\sqrt{x} + \sqrt{y} = \sqrt{x+y}$ (b) $\sqrt{x} \cdot \sqrt{y} = \sqrt{xy}$ (c) $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

9. For each of the numbers below determine, without using a calculator, the next whole number (i.e. the smallest whole number that is greater than the given number). Explain how you arrived at your answer.

   (a) $\sqrt{38}$ (b) $11 - \sqrt{23}$ (c) $\dfrac{\sqrt{119}}{\sqrt{43}}$

10. Are the following statements always true, sometimes true, or never true?

    (a) If $\frac{a}{b} < \frac{c}{d}$, then $\left(\frac{a}{b}\right)^2 < \left(\frac{c}{d}\right)^2$.

    (b) If $\frac{a}{b} < \frac{c}{d}$, then $\left(\frac{a}{b}\right)^{-1} < \left(\frac{c}{d}\right)^{-1}$.

10 Decimals

10.1 Place Value Continued

Back in the Fenland chapter one of the main focuses was to get a deep understanding of place value and the role it plays in computations. The key to understanding place value is the ability to work with the exchange rate. In Fenland, the exchange rate was five (which we called fen) to one. In other words, it took five ones to make a fen, five fens to make a fefen, etc. Of course, in the U.S. numeral system the exchange rate is ten to one. It takes ten ones to make a ten, it takes ten tens to make a hundred, it takes ten hundreds to make a thousand, etc. We can continue in this way ad nauseam adding more and more digits to the left of the number.

We now extend that place value system to the right as well. Suppose we want to put a digit to the right of the ones place. Well, the exchange rate must still be ten to one. Therefore whatever value that spot has, it must take ten of them to make a one. In other words, $10\times?$ must equal 1. Do we know any numbers that we could multiply by 10 and get 1? Of course we do, we just spent several pages learning about fractions! The question mark must be $\frac{1}{10}$. In other words, the spot to the right of the ones place must be tenths. In a similar manner, the spot to the right of tenths must still have an exchange rate of ten to one. Therefore, in that spot we must satisfy $10\times? = \frac{1}{10}$. Solving this then, we see the question mark must be $\frac{1}{100}$, or hundredths. We can continue this on to the right forever now.

Example 10.1.1. What is the value of the 4 in the number 23.5348?

Answer. The four is in the thousandths place, so the value is $\frac{4}{1000}$.

Now that we have extended place values to the right forever, we do face a bit of notational problem. Prior to this section, when we wrote 354, for example, we knew the right most digit was the ones place, so we knew this number meant 3 hundreds, 5 tens, and 4 ones. However, now that we know there are actually place values to the right of the ones place, then 354 is ambiguous. Is the last digit on the right the ones place, the tenths place, the hundredths place...who knows? As you might suspect, the solution to this ambiguity is the decimal point. We place a dot just to the right of the ones place so there is no question what value each place has. Therefore when we see 35.4 we know this means 3 tens, 5 ones, and 4 tenths. When a number has nonzero digits to the right of the ones place we call that number a **decimal**. So 35.4 is an example of a decimal.

Although all we have done is extend the place value system we already know, it is probably still worth while to spend some time getting comfortable with this new way to represent numbers. We will start by comparing numbers. As always, we want to be sure we are only using what we know so far. In the case of decimals, all we know is the value of each place.

Example 10.1.2. Which is larger 0.3 or 0.5?

Answer. 0.3 means 3 tenths, so $0.3 = \frac{3}{10}$, and 0.5 means 5 tenths, so $0.5 = \frac{5}{10}$. Using our knowledge of fractions we know $\frac{5}{10}$ is larger than $\frac{3}{10}$, so 0.5 is larger than 0.3.

A quick note about the 0 that was put in the ones place above. Some people have very strong opinions about whether or not the 0 should be there. For example, we don't write 0345 when we mean 345. So we certainly could write .3 and it still means $\frac{3}{10}$. To be honest, it doesn't really matter. However, often times if we just lead with a decimal point it can often get lost, especially if work is being hand-written. For this reason, when a number is less than one, I will always place a 0 in front of the decimal point.

The above example is a very straightforward comparison problem that probably would have been answered correctly by students even if they had just started learning decimals. Moving forward, we will proceed through a few more examples that may seem pretty straight forward to you. However, as we go through the solutions notice the number of steps required to finally see the answer. There are usually subtleties that we don't recognize if we are not careful because of our already developed comfort with decimals, so as we are moving through this section and those to follow be sure to force yourself to build everything up from just knowing the place value system.

Example 10.1.3. Which is larger, 0.3 or 0.24?

Possible Solution. The decimal 0.3 means 3 tenths, so $0.3 = \frac{3}{10}$. The decimal 0.24 means 2 tenths and 4 hundredths, so $0.24 = \frac{2}{10} + \frac{4}{100}$. Using our knowledge of fractions we see that $\frac{2}{10} + \frac{4}{100} = \frac{2\times10}{10\times10} + \frac{4}{100} = \frac{24}{100}$. We want to compare this to $\frac{3}{10}$. Again using our knowledge of fractions, we know that $\frac{3}{10} = \frac{3\times10}{10\times10} = \frac{30}{100}$.

Therefore $0.3 = \frac{30}{100}$ and $0.24 = \frac{24}{100}$, and using our knowledge of fractions we know $\frac{30}{100}$ is larger than $\frac{24}{100}$. Thus 0.3 is larger than 0.24. □

A student who is first learning decimals may be surprised by this result because if they are ignoring the decimal point they see the number 3 and the number 24. They know 24 is larger, so may assume 0.24 is larger. An important aspect to notice with these examples is the direct relationship to our knowledge of fractions, and it is important that students see these connections. In particular, when comparing fractions we needed to get a common denominator.

Question 10.1.4. When working with decimals, what is analogous to getting a common denominator?

This is a good time to pause and talk about the naming of decimal numbers. With whole numbers we know that 354 means 3 hundreds, 5 tens, and 4 ones, but that is not the name of the number. When we see 354, we say three hundred fifty four. In other words, there is a difference between saying what the numeral means and saying the name of the number. For example, 0.24 means 2 tenths and 4 hundredths. But what do we say when we see 0.24? Well, sometimes we just say "zero point two four", but this would be like seeing 354 and saying three five four. Now, I'm not completely opposed to seeing 0.24 and saying "zero point two four", however we should also be aware of the actual name of this number. The name is important because it really does give meaning to the number rather than just saying the digits. Let's look at a few examples before continuing this discussion.

Example 10.1.5. Is 0.527 a rational number?

Possible Solution. In order to show 0.527 is a rational number we need to express this number in rational form, namely $\frac{a}{b}$ where a and b are integers.

$$
\begin{aligned}
0.527 &= \frac{5}{10} + \frac{2}{100} + \frac{7}{1000} \\
&= \frac{5 \times 100}{10 \times 100} + \frac{2 \times 10}{100 \times 10} + \frac{7}{1000} \\
&= \frac{500}{1000} + \frac{20}{1000} + \frac{7}{1000} \\
&= \frac{527}{1000}
\end{aligned}
$$

Since we were able to write 0.257 in rational form that means 0.257 is a rational number. □

Example 10.1.6. Express 0.0037 as a fraction in rational form.

Possible Solution.

$$
\begin{aligned}
0.0037 &= \frac{3}{1000} + \frac{7}{10000} \\
&= \frac{3 \times 10}{1000 \times 10} + \frac{7}{10000} \\
&= \frac{30}{10000} + \frac{7}{10000} \\
&= \frac{37}{1000}
\end{aligned}
$$

Therefore $0.0037 = \frac{37}{10000}$. □

Example 10.1.7. What is the decimal representation for the fraction $\frac{259}{1000}$?

Possible Solution. In order to see what decimal this, we have to know what digit is in the tenths place, the hundredths place, etc. Therefore we need to break our fraction down into tenths, hundredths, and thousandths.

$$
\begin{aligned}
\frac{259}{1000} &= \frac{200 + 50 + 9}{1000} \\
&= \frac{200}{1000} + \frac{50}{1000} + \frac{9}{1000} \\
&= \frac{200 \div 100}{1000 \div 100} + \frac{50 \div 10}{1000 \div 10} + \frac{9}{1000} \\
&= \frac{2}{10} + \frac{5}{100} + \frac{9}{1000} \\
&= 0.259
\end{aligned}
$$

Therefore the decimal representation of $\frac{259}{1000}$ is 0.259. □

Question 10.1.8. Suppose we have a decimal with 5 digits to the right of the decimal point. If we convert that decimal to a fraction in rational form (and don't reduce the fraction), what will the denominator be?

From the examples above and this last question, we see that the number of digits to the right of the decimal point and the denominator of the fraction we initially get upon converting are directly related. That means that if we have a decimal like 0.24, we know it is equal to $\frac{24}{100}$. It is this fraction equivalence that gives us the name of the decimal, so the name of the number 0.24 is "twenty four hundredths". Naming decimals in this way helps students make connections between decimals, fractions, and whole numbers.

We have been working strictly with decimals less than one, but the methods we are employing still work with decimals larger than one. The only difference for numbers bigger than one is that we have two different ways to write it as a fraction, namely either as an improper fraction or as a mixed number. For example, if we were to convert the decimal 2.57 to a fraction using the methods above, we would either get $\frac{257}{100}$ or $2\frac{57}{100}$. Therefore we technically could call 2.57 either "two hundred fifty seven hundredths" or "2 and fifty seventy hundredths". Although there is nothing wrong with the former name, the latter name is the standard way to name the deicmal.

We have taken some time to get comfortable with decimals and we are ready to begin operating with decimals and to explore them more deeply. To do this we will be relying heavily on the connection between decimals and fractions. At this point we no longer need to show step by step how to convert a decimal to a fraction. For example, when we see the decimal 0.123 we can immediately write down $\frac{123}{1000}$. Similarly if we have a fraction that has a power of ten in the denominator, then we can immediately write down the decimal. For example, if we have the fraction $\frac{79}{100}$ we can immediately write down 0.79. I like to call fractions whose denominators are a power of ten "decimal friendly" fractions because we can immediately write down their decimal equivalent.

10.2 Operating with Decimals

If we wanted to add, subtract, multiply, or divide two decimals we could of course just convert them to fractions and then perform the operations using our knowledge of fractions. However, it would be nice if we could find a way to add, subtract, multiply, or divide decimals without having to convert to fractions, so that will be the focus of this section.

Let's begin by thinking about how we added and subtracted whole numbers. Way back in earlier sections we showed that in order to add/subtract two whole numbers we could just add/subtract the numbers place value by place value making exchanges along the way. Since all we have done with decimals is extend the number of place values, then there should be no change to the way we add/subtract. We still want to add/subtract the numbers place value by place value making exchanges along the way, so that means there is really no new method we need to learn, we just need to make sure our place values are lined up and then we can just add/subtract as we did with whole numbers.

There is a small notational issue that we should address. With whole numbers, in order to insure that place values were lined up when adding or subtracting we stacked the numbers and made sure they were right justified. That way ones were above ones, tens above tens, etc.

Question 10.2.1. If we have two decimals that we want to add or subtract, how do we insure that their place values are lined up?

Let's now move on to multiplication. The standard multiplication algorithm for whole numbers was much more complicated than just lining up place values, so we will have to spend some time figuring out how multiplication with decimals works. We will have to go back to our knowledge of fractions and work there first. While going through the examples we should look for generalizations that would allow us to skip the intermediary steps of converting to and working with fractions.

Example 10.2.2. Compute 2.34×52.8.

Possible Solution. We will convert the decimals to fractions, perform the operations with the fractions, and then convert back to a decimal.

$$\begin{aligned} 2.34 \times 52.8 &= \frac{234}{100} \times \frac{528}{10} \\ &= \frac{234 \times 528^*}{100 \times 10} \\ &= \frac{123552}{1000} \\ &= 123.552 \end{aligned}$$

*Note: To compute the product 234×528 we just use the standard multiplication algorithm for whole numbers.

Therefore $2.34 \times 52.8 = 123.552$ □

Example 10.2.3. Compute 5.023×0.57.

Possible Solution. We will convert the decimals to fractions, perform the operations with the fractions, and then convert back to a decimal.

$$\begin{aligned} 5.023 \times 0.57 &= \frac{5023}{1000} \times \frac{57}{100} \\ &= \frac{5023 \times 57^*}{1000 \times 100} \\ &= \frac{286311}{100000} \\ &= 2.86311 \end{aligned}$$

*Note: To compute the product 5023×57 we just use the standard multiplication algorithm for whole numbers.

Therefore $5.023 \times 0.57 = 2.86311$ □

After working through these examples, and perhaps a few more, we should be able to start generalizing what is going on. Let's look at the last example. The digits we got for our answer were 286311. How did we get those digits? The decimal point was placed right after the 2. Why was the decimal point placed there? After thinking carefully about these two questions we should be able to answer and justify the following.

Question 10.2.4. When multiplying two decimals, how do we determine the digits that will be in the answer?

Question 10.2.5. When multiplying two decimals, how do we determine where to place the decimal point?

In answering the two questions above, we can now have a method that allows us to multiply two decimals without having to convert them to fractions. So our last order of business is division. Recall that long division with whole numbers was just a place value activity. For example, if we wanted to compute $5324 \div 4$, we first distribute the 5 thousands fairly into 4 groups, so each group gets 1 and we have 1 thousand left over. We exchange that 1 thousand for 10 hundreds and together with the 3 hundreds we already have, we now have 13 hundreds. We distribute those 13 hundreds, etc.

For the most part then if we are dividing a decimal by a whole number, we can just continue this place value activity. There are some new things arise because we are working with decimals, so we will address that in the examples as well. Let's first consider the example $5.324 \div 4$. We can do this problem exactly like we would have with whole numbers.

To compute $5.324 \div 4$, we are thinking of this as having 5 ones, 3 tenths, 2 hundredths, and 4 thousandths, and we want to distribute that fairly into 4 groups. We first take the 5 ones and put 1 one in each group. We want to be sure we know that it is a one that we just distributed, so when we put our 1 up top we want to be sure we place the decimal point right after the 1 to indicate the place value. After distributing the ones, we have 1 one left. This portion of the long division is shown below.

$$
\begin{array}{r|cccc}
 & 1. & 3 & 3 & 1 \\ \hline
4 & 5. & 3 & 2 & 4 \\
 & -\,4 & & & \\ \hline
 & 1 & & &
\end{array}
$$

We exchange that 1 one for 10 tenths. Together with the 3 tenths we already have, that gives us a total of 13 tenths. This portion of the long division is shown below.

$$
\begin{array}{r|cccc}
 & 1. & 3 & 3 & 1 \\ \hline
4 & 5. & 3 & 2 & 4 \\
 & -\,4 & & & \\ \hline
 & 1 & 3 & &
\end{array}
$$

We now have 13 tenths to distribute to the 4 groups. Each group gets 3 tenths, and we have 1 tenth left over. This portion of the long division is shown below.

$$
\begin{array}{rrrrrr}
 & & 1. & 3 & 3 & 1 \\ \cline{2-6}
4 & \big) & 5. & 3 & 2 & 4 \\
 & - & 4 & & & \\ \cline{2-3}
 & & 1 & 3 & & \\
 & - & 1 & 2 & & \\ \cline{3-4}
 & & & 1 & &
\end{array}
$$

We exchange the 1 tenth for 10 hundredths, and together with the 2 hundredths we already had that gives us 12 hundredths. We distribute 3 hundredths to each group. Leaving us with no hundredths. This portion of the long division is shown below.

$$
\begin{array}{rrrrrr}
 & & 1. & 3 & 3 & 1 \\ \cline{2-6}
4 & \big) & 5. & 3 & 2 & 4 \\
 & - & 4 & & & \\ \cline{2-3}
 & & 1 & 3 & & \\
 & - & 1 & 2 & & \\ \cline{3-4}
 & & & 1 & 2 & \\
 & & - & 1 & 2 & \\ \cline{4-5}
 & & & & 0 &
\end{array}
$$

Finally we still have 4 thousandths, so we distribute 1 to each group, leaving us with nothing left. This portion of the long division is shown below.

$$
\begin{array}{rrcccc}
 & & 1. & 3 & 3 & 1 \\ \cline{3-6}
4 & \big) & 5. & 3 & 2 & 4 \\
 & - & 4 & & & \\ \cline{2-3}
 & & 1 & 3 & & \\
 & - & 1 & 2 & & \\ \cline{3-4}
 & & & 1 & 2 & \\
 & & - & 1 & 2 & \\ \cline{4-5}
 & & & & 0 & 4 \\
 & & & - & & 4 \\ \cline{5-6}
 & & & & & 0
\end{array}
$$

Therefore $5.324 \div 4 = 1.331$. In this case, because we were dividing by a whole number nothing changed in the long division process. In other words, because we were dividing by a whole number the idea of distributing to 4 groups still worked. If we were dividing by 4.2 for example, then the idea of distributing to 4.2 groups is much less intuitive. Therefore, when dividing by a decimal we have some work to do. It is possible to make sense of 4.2 groups, and then we could proceed from there. However, there is another option that involves much less work.

In this book the importance of being flexible with numbers and arithmetic has been highlighted multiple times, and we are at a spot where thinking flexibly is of value. Let's consider the problem $1.617 \div 4.2$. As discussed above, thinking about this problem as distributing 1.617 into 4.2 groups is not very intuitive, so we would like a different way to think about this. In particular, we would prefer if we were dividing by a whole number rather than a decimal.

Question 10.2.6. Is there a division problem equivalent to $1.617 \div 4.2$ that would be easier for us to solve?

We are able to justify the answer to the question above precisely because we have the flexibility to interpret a division problem as a fraction. This allows us to use the ever-powerful Equivalent Fractions Property. Generalizing this idea, we see that we can take any problem for which we are dividing by a decimal and write an equivalent division problem that involves dividing by a whole number. In this way, we can then use our long division process as shown above.

Example 10.2.7. If we want to compute $6.8456 \div 3.91$, what equivalent division problem should we solve instead?

Answer. $684.56 \div 391$.

With this fix in play, it may appear that we have conquered all of this issues that could possibly arise with decimals, but we are not quite there yet. Let's consider the problem $5.327 \div 4$. We are dividing by a whole number, so we don't need to change the problem. Let's begin the long division process.

$$
\begin{array}{rrrrrr}
 & & 1. & 3 & 3 & 1 \\ \cline{3-6}
4 & \big) & 5. & 3 & 2 & 7 \\
 & - & 4 & & & \\ \cline{3-4}
 & & 1 & 3 & & \\
 & - & 1 & 2 & & \\ \cline{4-5}
 & & & 1 & 2 & \\
 & & - & 1 & 2 & \\ \cline{5-6}
 & & & & 0 & 7 \\
 & & & - & & 4 \\
 & & & & & 3
\end{array}
$$

Unlike our long division above, when we got to the end of our decimal we still have something left over. Now, with whole numbers, we were often left with a remainder other than 0. So if the problem were whole numbers, namely $5327 \div 4$, then we would say the answer is 1331 r. 3. But that remainder of 3 was actually 3 ones. In our problem with the decimals, the 3 that is left is not 3 ones but 3 thousandths. Therefore if we were going to write this as a remainder we have to write something like this: $5.327 \div 4 = 1.331$ r. 0.003. Although that is mathematically correct we normally don't write answers to decimal division problems in that way.

Question 10.2.8. Why, if we were only working with whole numbers, would we have to stop at this point in the long division, but since we are working with decimals we can continue?

Notice that in answering the question above, we can also apply this idea to whole numbers. For example, consider the whole number division problem $134 \div 5$. If we were only working in the whole numbers, our answer to this division problem would have to be 26 r.4., but with decimals we can go further.

Example 10.2.9. Compute $134 \div 5$.

Possible Solution.

$$
\begin{array}{rrrrrr}
 & & & 2 & 6. & 8 \\ \cline{3-6}
5 & \big) & 1 & 3 & 4. & 0 \\
 & - & 1 & 0 & & \\ \cline{3-4}
 & & & 3 & 4 & \\
 & & - & 3 & 0 & \\ \cline{5-6}
 & & & & 4 & 0 \\
 & & & - & 4 & 0 \\
 & & & & & 0
\end{array}
$$

Therefore $134 \div 5 = 26.8$. □

The above example, again shows the flexibility in mathematics. The answer to the division problem $134 \div 5$ can be expressed in three different ways. Firstly, we could say the answer is 26 r.4. If we prefer to answer in fractions, then the answer is $26\frac{4}{5}$. Finally, as we saw above, if we prefer the answer in decimals we have 26.8. It is worth noting that for each of these answers we could think of a real life scenario for which that answer is the preferred choice. What would your scenarios be?

We can now bring this section to a close because we are able to add, subtract, multiply, or divide any two decimals. Looking back over this section, we will see that because we spent a great deal of time understanding operations with whole numbers and fractions, we only had to make adjustments here and there to apply them to decimals. The connection between understanding decimals and understanding fractions and whole numbers cannot be overstated.

10.3 Powers of Ten and Decimals

Powers of ten play a special role in our number system since our exchange rate is 10 to 1. In this section we will explore the interaction of powers of ten and decimals. We have already explored powers of ten in the exercises for the two exponent sections. In particular, we know that if n is a natural number, then 10^n is equal to 1 followed by n zeros. For example, 10^5 is a 1 with 5 zeros, so $10^5 = 100000$. We also know that 10^{-n} is equal to $\frac{1}{10^n}$. For example, $10^{-5} = \frac{1}{10^5} = \frac{1}{100000}$. Let's begin our exploration of how powers of ten behave within the set of decimals.

Consider the decimal 0.0000000000000000000032. This is a perfectly good decimal, but boy oh boy is it a long one. We can of course convert it to a fraction, but it is just as bad: $\frac{32}{10000000000000000000000}$. I think we can all agree that we really don't want to have to write out all of those zeros. It sure would be nice if we could express this number without having to write so much...

Notice that when we converted the long decimal to a fraction the denominator ended up being a 1 followed by 22 zeros. Therefore the denominator is just 10^{22}. We can now rewrite the very long decimal.

$$\begin{aligned} 0.0000000000000000000032 &= \frac{32}{10000000000000000000000} \\ &= 32 \times \frac{1}{10000000000000000000000} \\ &= 32 \times \frac{1}{10^{22}} \\ &= 32 \times 10^{-22} \end{aligned}$$

Wow! We were able to express a very long decimal (one that took 23 digits to write) in a much shorter way, namely 32×10^{-22} (which only took 6 digits to write). Now that is efficiency!

Example 10.3.1. What decimal is equal to 245×10^{-6}?

Possible Solution.

$$
\begin{aligned}
245 \times 10^{-6} &= 245 \times \frac{1}{10^6} \\
&= 245 \times \frac{1}{1000000} \\
&= \frac{245}{1000000} \\
&= 0.000245
\end{aligned}
$$

Therefore the decimal representation of 245×10^{-6} is 0.000245. □

Example 10.3.2. What decimal is equal to 3.7485×10^3?

Possible Solution.

$$
\begin{aligned}
3.7485 \times 10^3 &= \frac{37485}{10000} \times 1000 \\
&= \frac{37485000}{10000} \\
&= \frac{37485000 \div 1000}{10000 \div 1000} \\
&= \frac{37485}{10} \\
&= 3748.5
\end{aligned}
$$

Therefore the decimal representation of 3.7485×10^3 is 3748.5. □

As we were working through the above two examples, we should be looking for patterns that we can generalize. In particular, what affect does multiplying by a power of ten have on a decimal. Let's start by recalling what affect multiplying whole numbers by positive powers of ten has. For example, we know that 23×1000 is equal to 23000. We sometimes state this in the following way, "when multiplying a whole number by 10^n we just add on n zeros". Although this statement is true, the reasoning behind why that is is not apparent at all, which makes generalizing it to a decimal difficult.

Rather than thinking about this as adding on zeros, let's think about it as a shift in place value. For example, if we consider the number 23, what place value is the 2 in? If we then multiply 23 by 10, we get 230. What place value is the 2 in now?

Question 10.3.3. Why does multiplying by 10 bump each digit up one place value?

Generalizing the idea above, we could justify that multiplying by 100 would bump each digit up two place values. Since the solely relied on the exchange rate of 10 to 1, then there should be no difference in its effect on decimals. If we take any number and multiply it by 10, then each digit will bump up one place value. Consider 2.34×10. The 2 that is in the ones place will now be in the tens place, the 3 that was in the tenths place will now be in the ones place, and the 4 that was in the hundredths place will now be in the tenths place, so $2.34 \times 10 = 23.4$.

Unfortunately this is often one of the spots in the elementary school curriculum where students memorize a rule rather than understand what is going on. In particular, students often recall the following rule.

> If n is a positive integer, then multiplying a number by 10^n
> just moves the decimal point n places to the right.

Question 10.3.4. How can we justify the above rule in light of our discussion about place value above?

We now know the effect of multiplying by positive powers of ten on decimals, but what about the negative powers of ten? Looking back at Example 10.3.1 we see that a whole number times a negative power of ten can be converted to a decimal friendly fraction. That fraction determines where the decimal point must go. Imagine if the number in that example was 2.45 rather than 245, then in the third step we would get $\frac{2.45}{1000000}$. Unfortunately the fraction we would get does not have a whole number in its numerator, so the decimal friendly fraction conversion isn't as straight forward. Therefore when faced with a decimal there needs to be an intermediary step to alleviate this issue. To determine what is going on, let's just start by multiplying a decimal by 10^{-1} and see its affect.

Example 10.3.5. What decimal is equal to 3.28×10^{-1}?

Possible Solution.

$$\begin{aligned} 3.28 \times 10^{-1} &= 3.28 \times \frac{1}{10} \\ &= \frac{328}{100} \times \frac{1}{10} \\ &= \frac{328 \times 1}{100 \times 10} \\ &= \frac{328}{1000} \\ &= 0.328 \end{aligned}$$

In carefully analyzing the example above, we should be able to answer, with justification, the question below.

Question 10.3.6. What is the effect on a decimal if it is multiplied by 10^{-1}?

Question 10.3.7. How can we use the fact that we know the effect of 10^{-1} to determine the effect of 10^{-n}?

We have now reasoned through multiplying by 10^n and 10^{-n}, where n is a positive integer. The key word here being *reasoned*. That reasoning came from understanding the affect of these powers of ten on the place value of each digit. Therefore, although the rules students often learn in this setting is about *moving* the decimal point, reasoning has shown us that really it's the digits that are moving.

We now return to the issue discussed at the beginning of this section, namely the use of powers of ten to write very long decimals in a much shorter way. In that example, we were able to write 0.0000000000000000000032 as 32×10^{-22}. However, with our new found facility with powers of ten, there are multiple ways to write this number. For example, we could have done the following work instead.

$$
\begin{aligned}
0.0000000000000000000032 &= \frac{32}{10000000000000000000000} \\
&= 32 \times \frac{1}{10000000000000000000000} \\
&= 32 \times \frac{1}{10 \times 1000000000000000000000} \\
&= 32 \times \frac{1}{10} \times \frac{1}{1000000000000000000000} \\
&= \frac{32}{10} \times \frac{1}{1000000000000000000000} \\
&= \frac{32}{10} \times \frac{1}{10^{21}} \\
&= 3.2 \times 10^{-21}
\end{aligned}
$$

Therefore we could have written that long decimal as 3.2×10^{-21} as well.

Question 10.3.8. If we wanted to write this decimal using 10^{-23}, what would it be?

The desirability of writing long decimals more efficiently is exactly the motivation behind scientific notation. The idea is to use powers of ten to more efficiently write decimals. We have seen three ways to write that really long decimal above already, but only one of them is truly scientific notation. In all of these cases we wrote the number as "some number" $\times\ 10^{\text{some integer power}}$. Scientific notation requires the "some number" part to be a number greater than or equal to 1 and less than 10. Therefore the decimal 0.0000000000000000000032 written in scientific notation would be 3.2×10^{-21}.

Question 10.3.9. Without actually finding the product, how could you determine the number of digits to the right of the decimal point in the answer to the multiplication problem below.

$$(3.2 \times 10^{-21}) \times (4.8 \times 10^{-15})$$

Once you have answered the question above and are comfortable with the reasoning, here's a question that some might consider a *trick* question. In the question above, what if the multiplication problem was $(3.2 \times 10^{-21}) \times (4.5 \times 10^{-15})$ instead? Why might some consider this a trick question?

10.4 Terminating or Not?

In an effort to increase flexibility, we will spend this section more carefully exploring the connection between numbers written in fraction form versus decimal form. Because decimals are defined in terms of fractions (i.e. tenths, hundredths, thousandths, etc) we could take any of the decimals we have seen so far and convert them to a fraction. This process simply involved first writing down what the decimal means in terms of fractions and then add the fractions. It appears that converting a decimal to a fraction then is a simple matter. (Notice the use of the word "appears"! Your suspicions should be aroused...more on this later.) What about the other direction, though? If we have a fraction, can we convert it to a decimal?

We have already answered this question in the case of decimal friendly fractions. For example, if we are given the fraction $\frac{57}{1000}$, then we are comfortable writing down its decimal equivalent, namely 0.057. But what about fractions that are not decimal friendly? In other words, what about fractions whose denominators are not a power of 10?

Let's consider the fraction $\frac{1}{5}$. How could we rewrite this number as a decimal? Well, we know that $\frac{1}{5} = \frac{1\times 2}{5\times 2} = \frac{2}{10}$, and so $\frac{1}{5} = 0.2$. Of course, if we're feeling a little lazy we could just pull our calculators out and plug in $1 \div 5$ because we do know the equivalence of a division problem and a fraction. If we plug $1 \div 5$ into our calculator we do in fact get 0.2.

Let's look at a few more fractions, and in the interest of time, let's take the calculator route for now. Let's look at the fraction $\frac{1}{8}$. Plugging in $1 \div 8$, we get 0.125. Therefore $\frac{1}{8}$ written in decimal form is 0.125.

What about the fraction $\frac{1}{6}$. Plugging $1 \div 6$ into the calculator app on my phone, it returns 0.16666667. However, if I turn my calculator sideways it returns 0.1666666666666667. These two decimals are definitely not equal, so what is going on here? Because we have experience with decimals we know that what is most likely going on is that the decimal representation of $\frac{1}{6}$ is actually a repeating decimal, namely the decimal $0.666666\ldots$. Recall that we use a bar to indicate a repeating decimal, so we write $0.\overline{6}$ to mean $0.66666\ldots$. Are we sure that $\frac{1}{6}$ is equal to a repeating decimal. For example, maybe it ends after 20 digits and our calculator just can't show us that many. Now, in fact, it doesn't end after 20 digits, and it is in fact a repeating decimal. We will address how we know for sure in the next two sections, but it is important to recognize that we have not yet convinced ourselves of this fact.

Notice that in both cases above we started with a fraction, but in one case we got a decimal three digits long and in the other case we got a decimal that went on forever. A decimal that has a finite number of nonzero digits is called a **terminating decimal**. A more friendly way to describe this is that a terminating decimal is a decimal that actually stops. A decimal that has infinitely many nonzero digits is called a **non-terminating decimal**. In other words, a non-terminating decimal goes on forever. From above, 0.125 is an example of a terminating decimal whereas $0.66666\ldots$ is an example of a non-terminating decimal.

Question 10.4.1. Notice that the official definition for terminating given above is worded somewhat strangely. Why is it stated that way? In other words, what's up with the "finite number of nonzero digits" business?

We have been using our calculators to convert a fraction to a decimal. If we are going to use our calculator to find these decimals, how do we know if the decimal is terminating or not? Maybe it stops a few places past where our calculator shows, in which case it would be a terminating decimal. Or maybe it really does go on forever. Now, we may be able to make a fairly good guess by what we see in our calculator as we did with $\frac{1}{6}$. Because we saw all those 6's, we were pretty sure the decimal was repeating. But consider this: $\frac{1}{230}$. If we plug $1 \div 230$ into our calculator we get .004347826086957 (depending on how many digits your calculator shows). This unfortunately isn't very useful. We don't see any clear set of numbers being repeated. How do we know if that's all the digits or not? Does it stop there? A few more spaces further? Or not at all? Aaaargh! So many questions!

Clearly we have some work to do. The question: What sorts of fractions give rise to terminating decimals and what sorts of fractions give rise to non-terminating decimals? In this section we will explore ideas crucial to answering this question.

In the previous section, we used division to convert a fraction to a decimal. For example, to convert $\frac{1}{8}$ to a decimal we plugged $1 \div 8$ into our calculator. However, we soon found that plugging these division problems into our calculator caused potential confusion because we could not tell if our calculator was showing us everything. In particular, we didn't know if the decimal was terminating or not. Because converting by division has caused some problems for us, we are going to avoid that method for a while. Perhaps some enlightenment will come from looking at these conversion problems using a different method.

Recall that the very first fraction we converted to a decimal in this section was $\frac{1}{5}$, and we actually converted it in two different ways. One way was of course the division method, but the other way was to just change the fraction to something we recognized as a decimal. In particular, we used our knowledge of fractions to rewrite $\frac{1}{5}$ as $\frac{2}{10}$, which is a decimal friendly fraction. That means if we can take our given fraction and find a decimal friendly fraction equivalent to it, then we should be good to go.

Example 10.4.2. Express $\frac{39}{50}$ as a decimal (without computing $39 \div 50$).

Possible Solution. Looking at the denominator of 50, we see that we can multiply this by 2 to get 100. This will give us the decimal friendly fraction we need.

$$\frac{39}{50} = \frac{39 \times 2}{50 \times 2} = \frac{78}{100} = 0.78$$

Therefore $\frac{39}{50}$ expressed as a decimal is 0.78. □

Example 10.4.3. Express $\frac{23}{40}$ as a decimal (without computing $23 \div 40$).

Possible Solution. We first note that there is no whole number we can multiply 40 by to get 100. However, $40 \times 25 = 1000$, and so this will give us a decimal friendly fraction.

$$\frac{23}{40} = \frac{23 \times 25}{40 \times 25} = \frac{575}{1000} = 0.575$$

Therefore $\frac{23}{40}$ expressed as a decimal is 0.575. □

It would be helpful to get a few more examples under our belt so that we can start to generalize. For each of the fractions below, find (if possible) a decimal friendly equivalent fraction.

$$\frac{5}{16} \qquad \frac{99}{200} \qquad \frac{15}{18} \qquad \frac{213}{625} \qquad \frac{8}{22}$$

After studying the above fractions there are two things that we should address. First of all, you should have found that for some of the fractions you were not able to find an equivalent decimal friendly fraction. Second of all, some of the fractions above might have required a fair amount of guess and check work before you were able to find the necessary equivalent fraction.

Question 10.4.4. What types of fractions do not have a decimal friendly equivalent fraction?

Question 10.4.5. For fractions that do have a decimal friendly equivalent fraction, how can we efficiently find that decimal friendly fraction?

After answering the two questions above, we should be able to look at a fraction and determine whether or not it is equivalent to a decimal friendly fraction. Moreover, if we happen to be working with a fraction that is equivalent to a decimal friendly fraction, then by analyzing our answer to Question 10.4.5 we should be able to also come up with a way to tell how long the decimal expression for that fraction will actually be. When you are ready, try your hand at answering the following two questions for each of the fractions given below. The key here is that you should be able to answer these questions *without* finding the decimal friendly fraction or the decimal expression for the fractions.

$$\frac{3}{8} \qquad \frac{9}{14} \qquad \frac{31}{64} \qquad \frac{81}{98} \qquad \frac{99}{125} \qquad \frac{71}{80} \qquad \frac{23}{30}$$

1. Is the fraction equivalent to a decimal friendly fraction?
2. If the fraction is equivalent to a decimal friendly fraction, how long will the decimal representation of the fraction be?

After studying the fractions above, we should have a good handle on when a fraction is equivalent to a decimal friendly fraction. However, we should recall the big question we are trying to answer, namely what sorts of fractions give rise to terminating decimals and which don't. So what do decimal friendly fractions have to do with terminating decimals?

We know from the definition of place value that any terminating decimal can be written as a fraction by adding each of the place value fractions up. If we do not reduce that fraction, then the denominator will be 10 or 100 or 1000 or etc. depending on what place value the last digit was in. For example, consider the decimal 0.1234. The last digit is in the ten thousandths place and so this decimal is equal to a fraction with denominator 10000. In particular, $0.1234 = \frac{1234}{10000}$. Therefore we see that *every* terminating decimal is equal to a decimal friendly fraction.

Now suppose we have some fraction whose decimal representation is a terminating decimal, then that decimal must be equal to a decimal friendly fraction. But that means the original fraction must be equal to a decimal friendly fraction as well. In other words, being equal to a decimal friendly fraction is equivalent to being equal to a terminating decimal. Therefore if a fraction is not equivalent to a decimal friendly fraction then the decimal representation of that original fraction will not be a terminating decimal.

That means our answer to Question 10.4.4 tells us exactly which fractions will not be represented by a terminating decimal, and therefore must be represented by a non-terminating decimal. Moreover, we know how to tell when a fraction is equivalent to a decimal friendly fraction which in turn tells us that fraction will be represented by a terminating decimal. Therefore we have answered the big question! We can look at a fraction and know whether or not it will be represented by a terminating decimal *without* actually finding the decimal.

Question 10.4.6. Which of the following fractions will have a terminating decimal representation?

$$\frac{87}{110} \qquad \frac{59}{160} \qquad \frac{175}{432} \qquad \frac{140}{1287}$$

There is a small subtlety that we have not addressed. Consider the fraction $\frac{3}{6}$. Will the decimal representation of this fraction be terminating or not? At first glance, you might say it will be represented by a non-terminating decimal because the denominator has a 3 in its prime factorization. However, we know that $\frac{3}{6} = \frac{1}{2} = 0.5$, which is a terminating decimal. This shows that we need to be careful. In this example, the 3 that we thought would cause a problem ended up going away once we reduced the fraction. This should be a lesson to us that all of the claims we made above about how to tell terminating or not relied on the fact that the fraction was reduced.

We are now at the point that we can tell exactly when a fraction will be represented by a non-terminating decimal. For example, the decimal representation of $\frac{4}{7}$ will be non-terminating. If we plugged $\frac{4}{7}$ into a calculator that shows enough digits we would get 0.571428571428571, but of course we know this is not all of the decimal because we know $\frac{4}{7}$ will be represented by a non-terminating decimal. If we look at the digits in the calculator though, there does seem to be a pattern. In particular, it looks like the 571428 part is repeating, so it appears $\frac{4}{7}$ is equal to $0.\overline{571428}$. But are we sure? How could we determine that it definitely is a repeating decimal?

Before we explore that problem however, we need to be careful with our language again. We are throwing around the words non-terminating decimal and repeating decimal, so we need to be clear about the meaning of these words. Non-terminating decimal has already been defined; it is a decimal that goes on forever. The decimal $0.3333333\ldots$ is an example of a non-terminating decimal. However, it is also an example of a repeating decimal. A **repeating decimal** is a decimal that has a specific string of digits repeated infinitely. Since repeating decimals do go on forever, then a repeating decimal is just a specific type of non-terminating decimal. So are there non-terminating decimals that are not repeating decimals? Yes! For example, consider the decimal $0.12112111211112\ldots$. This decimal goes on forever, if we continue the pattern we see, but it is *not* a repeating decimal because it does not have the same string of digits repeated over and over. Therefore this decimal is an example of a non-terminating decimal that is not a repeating decimal.

But wait, that means we have another question to answer. We can tell when a fraction will be represented by a non-terminating decimal, but now the question is: what type of non-terminating decimal will it be? In other words, given a fraction that is represented by a non-terminating decimal, how can we tell if the decimal will be repeating or not? Let's keep this question in mind as we explore the original example in this section, namely that of the fraction $\frac{4}{7}$. We are now going to return to the method of division to work with these fractions, so to convert this fraction to a decimal we need to compute $4 \div 7$.

Question 10.4.7. Compute $4 \div 7$ by hand using long division. At what point in the long division process do you realize the decimal will repeat?

Considering our answer to the previous question, let's explore another example. Let's consider the fraction $\frac{3}{13}$. We are eventually going to compute $3 \div 13$ to determine the decimal representation, but before we actually do that computation let's think through some questions.

Question 10.4.8. Suppose we are dividing some number by 13. What numbers could we possibly get for a remainder?

Question 10.4.9. Without doing the long division for $3 \div 13$, explain how you know that you will not get a remainder of 0 at any step in the long division process?

Question 10.4.10. Without doing the long division for $3 \div 13$, determine how many possible numbers there are that could be a remainder at some point in the long division process?

Question 10.4.11. How does answering the three questions above help us determine whether or not $\frac{3}{13}$ will be repeating?

Okay, now actually do the long division. Did the long division process take as many steps as you expected before you saw a repeat? You should have found that $\frac{3}{13} = 0.\overline{230769}$. In a repeating decimal, the string of digits that are repeated is called the **repetend**. So in the decimal representation for $\frac{3}{13}$ the repetend is 230769. In other words, the repetend was 6 digits long. In answering the questions above, you should have expected to see a repeat by the thirteenth step in the long division process. In this case, it happened before the thirteenth step.

If we revisited the questions above for $\frac{4}{7}$, we should expect to see a repeat by the seventh step of the long division process. Look back at your long division for $\frac{4}{7}$. That long division did take all seven steps before a repeat occurred.

Question 10.4.12. Without actually finding the decimal, how do you know that the decimal representation of $\frac{5}{11}$ will be a repeating decimal.

We can generalize our answer to the question above to apply to any rational number. Let's consider the fraction in rational form $\frac{a}{b}$. In the last section we were able to determine whether the corresponding decimal was terminating or non-terminating. Suppose we are in the situation where the corresponding decimal for $\frac{a}{b}$ is non-terminating. Then, using an argument similar to the one used in answering Question 10.4.12, we know that the decimal must be a repeating decimal. In other words, there are no rational numbers whose decimal representation is non-terminating and non-repeating. Therefore the moral of this section is the following statement: *The decimal representation of a rational number* $\frac{a}{b}$ *is either a terminating decimal or a repeating decimal.*

10.5 Exercises

1. With the introduction of decimals, Max is very confused about the role zeros play in a number. For example 3 and 30 are not equal, but 3 and 03 are. Similarly 0.3 and .3 are equal, 0.3 and 0.30 are equal, but 0.3 and 0.03 are not. Explain to Max when and why introducing zeros affects the value of the number.

2. (a) Show how a student who knows what decimals are but has not been taught any shortcuts for multiplying decimals could figure out what 2.45×2.8 equals. Show and justify each step.

   (b) The standard algorithm for multiplying decimals says:

   Step 1: Multiply the two numbers as if they were whole numbers.

   Step 2: Count the total amount of places behind the decimal point in the factors, and place the decimal point in your answer so that there are that many places behind the decimal point.

   - Indicate where Step 1 appears in your work for Problem 2a.
   - What part of your work for Problem 2a justifies the placement of the decimal point as instructed in Step 2? Be sure to explain how the part you indicated justifies the placement of the decimal point.

3. Determine whether the following statements are always true, sometimes true, or never true. Justify your response.

   (a) When multiplying two decimals, if you move the decimal point in both numbers one place to the left and then multiply them, you will get the same answer as if you multiplied the original decimals.

   (b) When dividing two decimals, if you move the decimal point in both numbers one place to the left and then divide them, you will get the same answer as if you divided the original decimals.

   (c) When adding two decimals, if you move the decimal point in both numbers one place to the left and then add them, you will get the same answer as if you added the original decimals.

4. Consider the problem $2.457 \div 8.32$. When trying to find this quotient by hand we are told to change the problem to $245.7 \div 832$ and then do the long division.

   (a) Why are we told to do it that way?

   (b) Why is that way valid?

5. Max thinks he has discovered another way to compute the quotient $2.457 \div 8.32$. He describes his method for this example below.

   > In order to compute $2.457 \div 8.32$ I just computed $2457 \div 832$ first. Then because the divisor has one less digit behind the decimal point than the dividend, I move the decimal point in the answer one place to the left.

(a) If you applied Max's method to $34.513 \div 2.8$, what would you need to do?

(b) If you applied Max's method to $3451.3 \div 2.83$, what would you need to do?

(c) Is Max's method valid? Justify your response.

6. Max says to you that he thinks $.36 > .4$ because $36 > 4$. Think of what you could ask Max that would help him to see his mistake. Please record this as a back and forth conversation that allows Max to recognize his error and correct it, without being told.

7. Suppose $\frac{a}{b}$ is a rational number in simplest form. Further suppose that the factorization of b has only twos and fives.

   (a) We came to the conclusion that in this case the decimal representation of $\frac{a}{b}$ will be a terminating decimal. Provide a summary of the reasoning behind this conclusion.

   (b) Explain how to determine how long the terminating decimal will be without actually finding the decimal.

8. Each of the fractions below, if converted to a decimal, would be a terminating decimal. That means the given fraction must be equivalent to a fraction whose denominator is a power of 10 (i.e. 10 or 100 or 1000 or 10000 etc).

   Determine, without guessing and checking, what multiplier must be used to convert it to a fraction whose denominator is a power of 10. Explain or show enough work, so that it is clear that you did not guess and check.

   Once you have determined the multiplier, use it to convert the fraction to a decimal.

   (a) $\frac{3}{64}$ (b) $\frac{9}{40}$ (c) $\frac{9}{3200}$ (d) $\frac{11}{625}$

9. Suppose $\frac{a}{b}$ is a rational number in simplest form. Further suppose that the factorization of b has at least one prime number that is not a two or a five.

   (a) We came to the conclusion that in this case the decimal representation of $\frac{a}{b}$ will not be a terminating decimal. Provide a summary of the reasoning behind this conclusion.

   (b) Furthermore we came to the conclusion that the decimal representation will in fact be a repeating decimal. Provide a summary of the reasoning behind this conclusion.

10. Without actually computing the answer to the division problem determine whether or not you will eventually get a remainder of zero in the long division process. Justify your response.

    (a) $1421957 \div 32768$ (b) $16000000 \div 98304$

11. For each of the problems below assume you are using a calculator that only shows up to 8 digits. Please answer each of the questions **without** actually finding the correct decimal.

(a) When using your calculator to convert $\frac{5}{512}$ to a decimal, the calculator returns .00976562. Explain how you know your calculator is not giving you the actual decimal representation equal to $\frac{5}{512}$.

(b) When using your calculator to convert $\frac{2}{7}$ to a decimal, the calculator returns .28571429. Explain how you know there is not actually a 9 in the decimal representation of $\frac{2}{7}$.

12. Determine whether the decimal representations of the following numbers are terminating, repeating, or neither. For any that are terminating, find their decimal representation by converting the fraction to a decimal friendly fraction. For any that are repeating or neither, briefly explain why.

(a) $\frac{63}{80}$ (b) $\frac{123}{175}$ (c) $\frac{11}{625}$

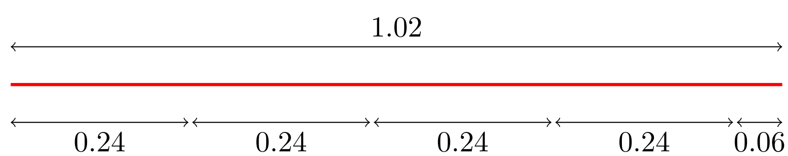
13. For each of the problems below give a brief explanation of how you came up with your answer.

(a) Find a fraction in rational form that lies between 0.23 and 0.24.

(b) Find a repeating decimal that lies between 0.24 and 0.25.

(c) Find a repeating decimal that lies between $\frac{3}{7}$ and $\frac{5}{12}$.

14. Note that $1.02 \div 0.24 = 4.25$, and consider the diagram below.



(a) What in the diagram represents the whole number part of the quotient (the 4)?

(b) What in the diagram represents the fractional part of the quotient (the .25)?

11 Rational and Irrational Numbers

11.1 Rational Numbers Revisited

Recall that a rational number is a number that can be expressed in the form $\frac{a}{b}$ where a is an integer and b is a non-zero integer. In Section **??**, we explored some unusual looking fractions and determined whether or not they were rational numbers. If we were able to write the unusual looking fractions in rational form, then we knew that they were indeed rational numbers even though they looked weird. Now that we have a new way to express numbers, namely decimals, we need to revisit the question of rational or not.

Example 11.1.1. Is $\frac{2.5}{3}$ a rational number?

Possible Solution. Since 2.5 is not an integer, this fraction is not currently in rational form. Let's use our knowledge of equivalent fractions to find an equivalent fraction that is in rational form.

$$\frac{2.5}{3} = \frac{2.5 \times 10}{3 \times 10} = \frac{25}{30}$$

Since $\frac{2.5}{3} = \frac{25}{30}$, then we have written $\frac{2.5}{3}$ in rational form. Therefore $\frac{2.5}{3}$ is a rational number. □

Question 11.1.2. Is $\frac{3.2}{5.13}$ a rational number?

We worked very hard in the previous two sections and figured out a lot of interesting things about converting rational numbers to decimals. However, we want to be sure we are not overstating what we have discovered so far. In the last section we found that when we converted a number in rational form, $\frac{a}{b}$, to a decimal the result was either terminating or repeating. To be clear then, at this point we have proven that all rational numbers are terminating or repeating decimals, but we have *not* addressed the reverse. In other words, are all terminating decimals rational numbers? Are all repeating decimals rational numbers? The former question we have already answered, because we know any terminating decimal is equal to a decimal friendly fraction. Therefore all terminating decimals are rational numbers. The answer to the latter question will take a little more work, so let's get started.

Let's consider the repeating decimal $0.\overline{7}$, so $0.77777\ldots$. If we want to determine whether or not this decimal is a rational number, then we need to figure out if it is equal to a fraction in rational form. Let's see what we can do.

We are looking for a rational fraction that is equal to this decimal. Let's use a box to represent that unknown fraction we are looking for. What we do know is that we want the box to be equal to our decimal. In other words we want the following.

$$\square = 0.77777\ldots$$

Let's multiply everything by 10, and see where it gets us.

$$10 \times \square = 10 \times 0.77777\ldots$$

$$10 \times \square = 7.77777\ldots$$

Let's now subtract our original equation from that last equation.

$$10 \times \square - \square = 7.77777\ldots - 0.77777\ldots$$

Question 11.1.3. How can we proceed from here to determine what fraction must be hiding in the box?

Since we are in fact able to find a rational fraction that is equal to $0.\overline{7}$, then that means $0.\overline{7}$ is a rational number. We could use exactly the same method to show that the decimal $0.\overline{4}$ is equal to the fraction $\frac{4}{9}$, and hence $0.\overline{4}$ is a rational number. Although we could use similar ideas, the *exact* same steps won't quite work to find the fraction equal to $0.\overline{45}$.

Question 11.1.4. What would need to be altered in the work above in order to successfully find a rational fraction equal to $0.\overline{45}$?

With enough practice, we should feel comfortable finding a fraction equal to a repeating decimal like $0.\overline{\text{some stuff}}$. Unfortunately not all repeating decimals look like that. For example, consider the decimal $0.3\overline{45}$, so $0.345454545\ldots$.

Question 11.1.5. How could we alter our method to accommodate a decimal like $0.3\overline{45}$?

With the development of this last method, we now can in fact take any repeating decimal and find a rational fraction equal to it. Because we can write any repeating decimal in rational form, that means all repeating decimals are rational numbers. And so we have come full circle. In the previous chapter, we showed that all rational numbers when converted to a decimal are either terminating or repeating. And we have now shown that all terminating decimals and all repeating decimals are in fact rational numbers. In other words, the set of rational numbers and the set of terminating and repeating decimals are the same set of numbers.

11.2 Irrational Numbers

Now that we know that only terminating and repeating decimals are rational numbers, that means we have examples of irrational numbers. Any decimal that is non-terminating but not repeating is an example of an irrational number. So the decimal, $0.12112111211112\ldots$ is an irrational number. You have probably heard that number π is also an irrational number, but unfortunately we cannot prove that in this class. It may seem that irrational numbers are just weird numbers, but it turns out that is not always true. There is a whole set of numbers that we use all the time that we will be able to show are in fact irrational numbers. We'll get there eventually, but first we need to do a little preparatory work.

Question 11.2.1. Suppose we have a number that has three 5's in its prime factorization. How many 5's will be in the prime factorization of the number squared?

Question 11.2.2. Suppose a is a number that has three 5's in its prime factorization. How many 5's will be in the prime factorization of the number $20a^2$?

Question 11.2.3. Suppose a is a positive integer. Are the following statements always true, sometimes true, or never true?

- The number a^2 has no 3's in its prime factorization.
- The number $9a^2$ has no 3's in its prime factorization.
- The number $9a^2$ has four 3's in its prime factorization.
- The number $9a^2$ has three 3's in its prime factorization.
- The number $9a^2$ has no 5's in its prime factorization.
- The number $9a^2$ has one 5 in its prime factorization.

Believe it or not, we are ready to show that there is a whole set of numbers that we use all the time that are irrational. We'll start with an example. Consider the number $\sqrt{5}$. Let's try to figure out whether or not we can write this number in rational form. If we could write it in rational form, then there would have to be natural numbers a and b so that $\sqrt{5} = \frac{a}{b}$. If we square both sides of this equation, then that would mean the equation $5 = \frac{a^2}{b^2}$ must be true. Multiplying both sides by b^2, then that means the equation $5b^2 = a^2$ must be true.

Working backwards that means that if we can somehow show that the equation $5b^2 = a^2$ cannot be true, then none of the equations that came before it can be true either. In particular, that would mean the equation $\sqrt{5} = \frac{a}{b}$ cannot be true. In other words, if we can show that the equation $5b^2 = a^2$ cannot possibly be true, then that means we cannot write $\sqrt{5}$ in rational form. And hence $\sqrt{5}$ must be irrational. So that is our goal, how could we argue that that equation can't possibly be true.

Question 11.2.4. How could we use the ideas learned in our preparatory work to justify why $5b^2 = a^2$ cannot possibly be true?

We should be able to generalize this method to show there are other square roots that are irrational. Could we use a similar type of argument to show that $\sqrt{3}$ is irrational? What about $\sqrt{45}$? or $\sqrt{225}$? or $\sqrt{200}$?

Question 11.2.5. If a is some positive integer, when will $\sqrt{a}$ be an irrational number?

We have been focusing on square roots, but we certainly could have focused on cube roots instead. What would have to be changed in our preparatory work in order to determine whether or not $\sqrt[3]{a}$ is irrational? What about fourth roots? fifth roots? etc.

11.3 Exercises

1. Convert the following decimals to fractions in simplest form. Please show all steps clearly.

   (a) $.\overline{753}$ (b) $.2\overline{38}$

2. Use the method outlined in the text to show that $\sqrt{7}$ is an irrational number.

3. For each of the numbers below, if we were to use the method outlined in the text to show the number is irrational, what prime number could be focused on in the method? Justify your response.

   (a) $\sqrt{26}$ (b) $\sqrt{99}$ (c) $\sqrt{72}$

4. For each of the expressions below a is some mystery integer. Answer, with justification, the following three questions for the number represented by each expression.

   - What is the minimum amount of 2's the number could have in its prime factorization?
   - Other than the minimum, give an amount of 2's the number could have in its prime factorization.
   - Give an amount of 2's larger than the minimum that the number will never have in its prime factorization.

   (a) $5a^2$ (b) $50a^2$ (c) $360a^2$ (d) $112a^2$

5. For each of the expressions below a is some mystery integer. Answer, with justification, the following three questions for the number represented by each expression.

   - What is the minimum amount of 2's the number could have in its prime factorization?
   - Other than the minimum, give an amount of 2's the number could have in its prime factorization.
   - Give an amount of 2's larger than the minimum that the number will never have in its prime factorization.

   (a) $5a^3$ (b) $4a^3$ (c) $2a^6$ (d) $24a^4$

6. Find an irrational number that lies between the two given numbers. Briefly explain how you know that the number you have found is irrational.

(a) 0.23 and 0.24

(b) $\frac{3}{5}$ and $\frac{4}{7}$

(c) $\sqrt{2}$ and $\sqrt{3}$

(d) π and $\pi + \frac{1}{100}$

7. Find a rational number and an irrational number between $23.98\overline{3499}$ and $23.98\overline{3500}$. Please briefly explain how you arrived at your answer and how you know the number is rational/irrational.

8. For each of the problems below please make it clear how you arrived at your answer.

   (a) Express as a fraction the distance between $\frac{1}{9}$ and 0.11.

   (b) Express as a fraction the distance between $\frac{1}{9}$ and 0.1111.

   (c) By just looking at your work for the above two problems, what do you think the distance between $\frac{1}{9}$ and 0.1111111111111111 will be?

9. Determine whether the following statements are always true, sometimes true, or never true? Justify your response.

   (a) A rectangle with an irrational perimeter will have a rational area.

   (b) A rectangle with a rational perimeter will have an irrational area.

10. Priyanka plugs $\sqrt{13}$ into her calculator and it spits out 3.60555. She originally thinks that $\sqrt{13}$ is equal to the decimal 3.60555, but then she is not sure. After looking at the 5's, she then thinks that perhaps $\sqrt{13} = 3.60\overline{5}$. Without doing any more calculations, explain to Priyanka how you know that $\sqrt{13}$ is not equal to 3.60555 nor $3.60\overline{5}$.

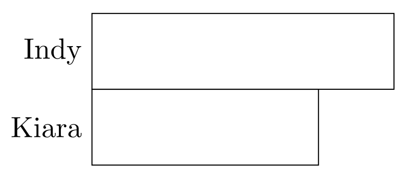
12 Ratios and Proportional Reasoning

12.1 Algebra Problems Before Algebra

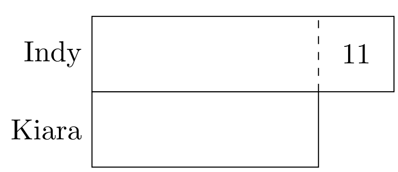
One of the most common places students struggle with math for the first time is in algebra. There are several reasons for this, but one of them is that algebra somehow seems drastically different from what the students have been doing. Therefore as teachers, we should find ways to involve the students in algebraic thinking well before they are introduced to the formalities of algebra. In this section we will explore a method to solving story problems that will help better prepare the students for algebra. Consider the story problem below.

> Indy has 11 more marbles than Kiara. Together they have a total of 73 marbles. How many marbles does Kiara have?

You probably recognize this as a standard algebra problem, so if it is really driving you crazy, go ahead and solve it using algebra. However, we want to explore how this problem could be solved by a fourth of fifth grader for example. In other words, without the formalities of algebra, but rather just algebraic thinking. We will do this by using a visual called a bar diagram. We will draw two boxes to represent the amount of marbles Indy has and the amount of marbles Kiara has. Already there is some thinking going on about this problem, because in making this drawing we now have to think about which box should be longer. Who has more? Well, Indy does so her box should be longer. Therefore we get the following picture.



We now have a visual representation of the scenario, so we should think about what other information can be added to this picture. Well, we know that Indy has 11 more marbles than Kiara, so we can represent that in the picture now.



Question 12.1.1. How can we proceed from here to determine how many marbles Kiara has?

12.2 Ratios vs. Fractions

We have all dealt with ratios before, and unfortunately that will be our downfall. The topic of ratios is an area where our knowledge, if we are not careful, can get in the way of really diving deep into understanding ratios from the beginner level. In other words, this is an area that is sometimes difficult to see from a new learner's point of view. Throughout this section we must continually force ourselves to only use what we are given, and not prior knowledge we have built up, in solving ratio problems.

In light of our conversation above, the most pressing issue for us to address is the fact that a ratio and a fraction are not the same thing. Let's consider the ratio 2 to 3. This means that for every 2 of "something" there will be 3 of "something else". Whereas the fraction $\frac{2}{3}$ means we have one "something", we break that something into 3 equal pieces and take 2 of those pieces. Therefore when students first learn ratios they in no way recognize them as being the same as a fraction. That idea is something that will need to be developed. For this reason, we will use the colon notation to represent a ratio until we get to the point where we understand why it is okay to use the fraction notation for a ratio. So when we mean the ratio 2 to 3, we will write $2 : 3$. We will only write $\frac{2}{3}$ if we actually mean the fraction two thirds.

We will use an example to help us think about ratios from a beginner's viewpoint. Suppose we have an artist, let's call her Ms. Pastel, and she thinks the perfect shade of orange paint is made by mixing 2 cups of yellow paint with 3 cups of red paint.

Question 12.2.1. If you mix 4 cups of yellow paint and 6 cups of red paint, will you get Ms. Pastel's perfect orange?

Question 12.2.2. If you mix 4 Tbsp of yellow paint and 6 Tbs of red paint, will you get Ms. Pastel's perfect orange?

In answering these questions, we get a better sense of what we mean by the ratio 2:3. We see that we can get the perfect orange paint as long as we use 2 parts yellow and 3 parts red. What the "part" actually is does not matter. The only thing that matters is that we keep the correct number of parts for each piece. Therefore we can now say that perfect orange is made by mixing yellow and red in a ratio of 2:3.

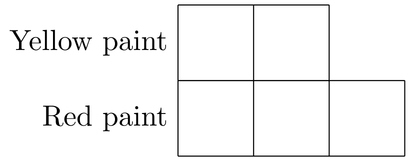
Example 12.2.3. If we mix 10 cups of yellow paint and 15 cups of red paint, do we get perfect orange?

Possible Solution. We know that perfect orange must come in a ratio of 2:3. Therefore if this is perfect orange, then the 10 cups must be made up of two "parts", and the 15 cups must be made up of 3 of those same "parts". Breaking 10 cups into 2 groups, we get 5 cups. Therefore a "part" must be 5 cups.

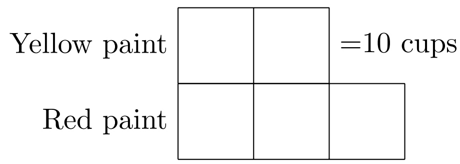
Now let's check that the red paint is in fact made up of 3 parts. If each part is 5 cups, then 3×5 gives us a total of 15 cups. Therefore the red is made up of 3 parts. Hence we will get perfect orange because the yellow to red ratio is 2:3. □

Question 12.2.4. If we mix 5 cups of yellow paint and 7 cups of red paint, will we get perfect orange? If so, what are the "parts"? If not, is the paint too red or too yellow?

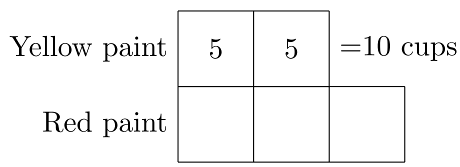
There is a very nice visual that helps students work with ratios like this. It is called a tape diagram. In a tape diagram we use equal size boxes to represent the parts, and then we can reason out what the value of each box (or part) must be. A tape diagram for Ms. Pastel's paint is shown below.



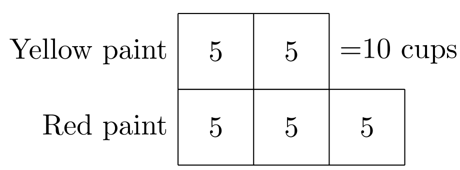
Let's revisit Example 12.2.3 to see how we can use the tape diagram to solve the problem. We know that there should be 10 cups of yellow paint. Since the top row of the diagram represents the yellow paint, then the top row must have a value of 10 cups.



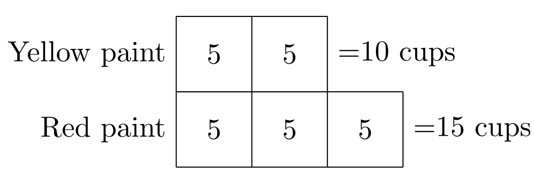
Therefore we need to fairly distribute those 10 cups into the two boxes. Since $10 \div 2 = 5$, then there must be 5 cups in each box. (To save space, we will leave off the units.)



Because all the parts have to have the same value, then each box must have a value of 5.



Totaling the row representing red, that means that to make perfect orange we would need 15 cups.

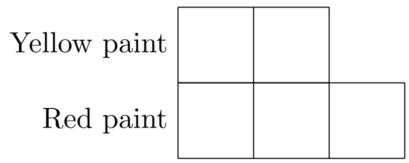


Therefore 10 cups of yellow and 15 cups of red do indeed make perfect orange.

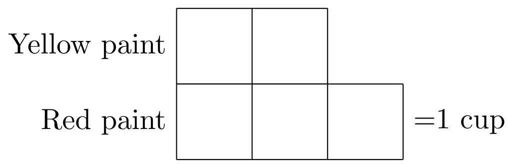
The above is just one example of how a student could use the tape diagram to solve the problem. You are encouraged to think about other ways students might use the tape diagram to get to the answer.

Example 12.2.5. You only have one cup of red paint. If you want to make Ms. Pastel's perfect orange, how much yellow paint do you need?

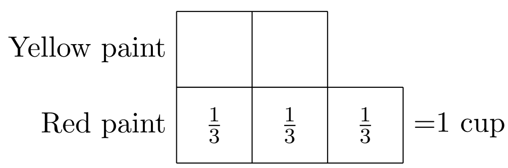
Possible Solution. We will use a tape diagram to solve this problem. The diagram representing this scenario is shown below.



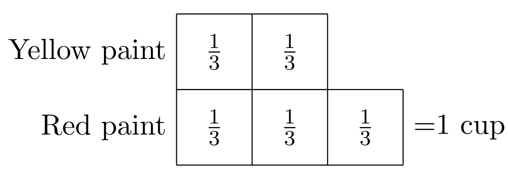
We only have 1 cup of red paint, so the value of the second row must be 1 cup.



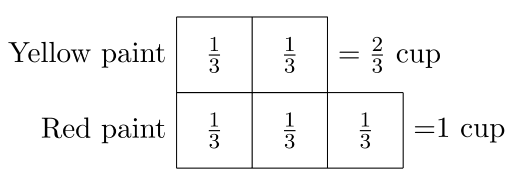
Distributing that 1 cup fairly into the three boxes, we see that the value of each box must be $\frac{1}{3}$.



All parts must be equal, so the value of the boxes in the top row must also be $\frac{1}{3}$.



That makes the top row have a value of $\frac{2}{3}$ of a cup.



Therefore if you only have one cup of red paint, then you will need to mix in $\frac{2}{3}$ of a cup of yellow paint to get perfect orange. □

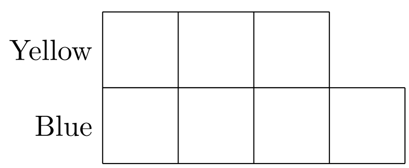
Notice that the actual fraction two thirds appeared in this 2:3 ratio problem, so we are beginning to see that there is a connection between ratios and fractions.

Question 12.2.6. How could we change the question in Example 12.2.5, so that the fraction $\frac{3}{2}$ appeared?

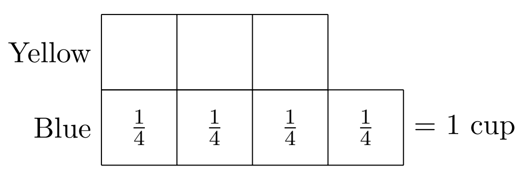
Let's consider a new scenario. Max and Macy are making green paint. Max likes his green paint to be mixed in a ratio of 3 yellows to 4 blues. Macy likes her green paint to be mixed in a ratio of 4 yellows to 5 blues.

Example 12.2.7. How many cups of yellow paint must Max add to 1 cup of blue paint in order to get his perfect shade of green?

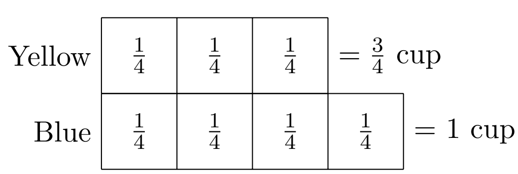
Possible Solution. Let's first set up the tape diagram for Max's paint.



Max has 1 cup of blue paint, so if we distribute that 1 cup fairly to the four boxes, the value of each box must be $\frac{1}{4}$.



Since all parts must have the same value, then the top row boxes must also have a value of $\frac{1}{4}$, which gives the top row a total value of $\frac{3}{4}$ of a cup.



Therefore we have found that Max must mix $\frac{3}{4}$ of a cup of yellow paint with his 1 cup of blue paint to make his perfect shade. □

Example 12.2.8. How many cups of blue paint must Macy add to 1 cup of yellow paint in order to get her perfect shade of green?

Possible Solution. Let's first set up the tape diagram for Macy's paint.

Yellow					
Blue					

Macy has 1 cup of yellow paint, so if we distribute that 1 cup fairly to the four boxes, the value of each box must be $\frac{1}{4}$.

Yellow	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$= 1$ cup	
Blue						

Since all parts must have the same value, then the bottom row boxes must also have a value of $\frac{1}{4}$, which gives the bottom row a total value of $\frac{5}{4}$ or $1\frac{1}{4}$ cups.

Yellow	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$= 1$ cup	
Blue	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$= 1\frac{1}{4}$ cups

Therefore we have found that Macy must mix $1\frac{1}{4}$ cups of blue paint with her 1 cup of yellow paint to make his perfect shade. □

Question 12.2.9. Whose shade of green is more bluish?

Although we have worked within a very specific scenario (that of mixing paint), we have developed a facility with ratios using only the new learner's background. Although tape diagrams are not the only way to solve problems involving ratios, it is an extremely valuable tool that serves to ground students' understanding of what a ratio is. This makes tape diagrams an excellent tool to use during the initial introduction of ratios. In addition it allows students to solve what would normally be considered algebra problems long before they are in algebra.

Lastly, we want to end this section by coming back to the ratio versus fraction discussion. In the examples above we saw how the fraction corresponding to the ratio can appear. In other words, within any ratio scenario, say $a : b$, there is a meaning within that scenario for the fraction $\frac{a}{b}$. That definitely shows a connection between ratios and fractions, but they are still different things, so it does not justify why we use fraction notation for ratios. To answer that comes down the the Equivalent Fractions Property. We know in fractions that $\frac{a}{b} = \frac{a \times n}{b \times n}$, so if we can show that ratios have the same behavior, then that could justify why we use fraction notation for ratios.

Question 12.2.10. If we have a scenario that upholds the ratio 2:3, does the scenario necessarily uphold the ratio 6:9?

We should be able to generalize our ideas above to justify that a ratio of $a : b$ also means a ratio of $a \times n : b \times n$, and vice versa. In other words, a ratio of $a : b$ is equivalent to a ratio of $a \times n : b \times n$. Therefore we see that as far as equivalence goes, ratios and fractions behave the same way.

12.3 Proportional or Not?

This section will include scenarios for which there is a proportional relationship and scenarios for which there is not a proportional relationship. This will be tied to the graphs of lines.

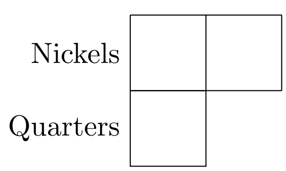
12.4 Proportional Reasoning Before Algebra

Many of the problems we will look at in this section can be solved using algebra, but we want to focus on what students can do without the formality of algebra. In addition, we want to explore how developing this proportional reasoning and the tools used to do so can help them with the algebra skills once they get into algebra.

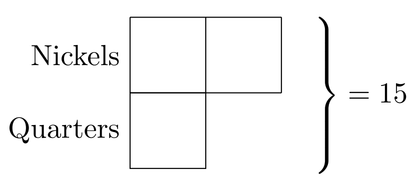
Let's consider a type of story problem that is commonly given in algebra courses. We will first solve this problem using a tape diagram and then compare it to the method expected in an algebra course.

> Yazmin has twice as many nickels as quarters. If she has a total of 63 coins, how many quarters does she have?

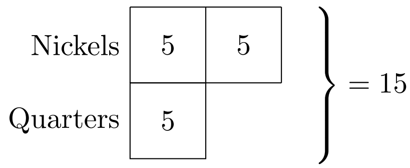
We first set up the tape diagram to represent this scenario. We have twice as many nickels as quarters, so we get the following diagram.



We know that the total number of all coins is 15, so the value of all three boxes together is 15.



Distributing 15 fairly to the three boxes, we see that each box must have a value of 5.



Therefore the total number of quarters is 5.

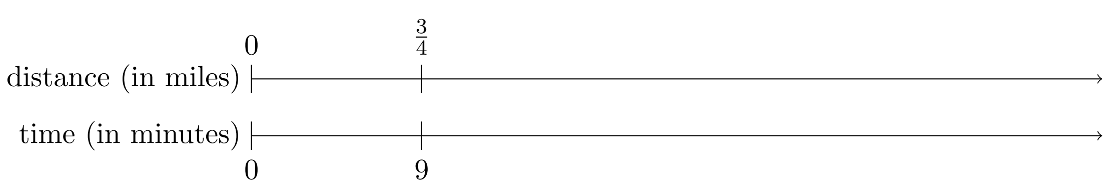
Let's now revisit this problem solving it using an algebraic equation. Let x represent the number of quarters. Since there are twice as many nickels as quarters, the number of nickels is $2x$. (Notice that if the student has the tape diagram in their head and see the x in the quarters box, then they will know that the number of nickels is $2x$.) Therefore the total number of coins is given by $x + 2x$, and we know this must equal 15, so $x + 2x = 15$. (Again, notice how this is supported by the tape diagram.) Therefore $3x = 15$, and to solve for x we divide by 3 (which is exactly what happened in the tape diagram) to get $x = 5$.

Therefore we see that a students ability to represent this visually can actually help them create the equation and help them see that the parts of their algebraic expression actually have meaning in the context of the problem. This makes algebra, which can sometimes seem rather abstract, a little more concrete.

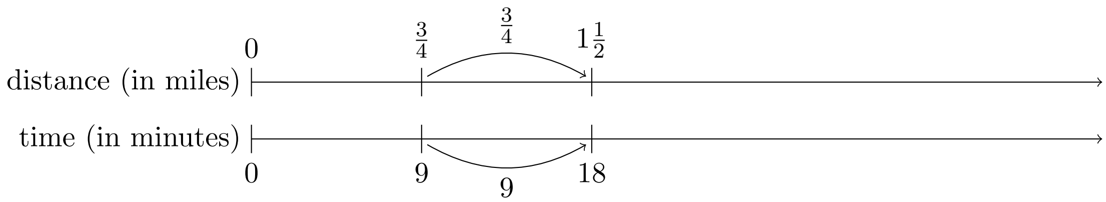
We now move on to exploring two more tools that can be extremely useful when solving proportional reasoning problems. They are double number lines and ratio tables. Again, the purpose of these tools, is to allow students to perform rather complicated proportional reasoning without the use of algebra. The emphasis here is on how these tools help the student *reason* about the problem, rather than memorize a specific method.

We first introduce double number lines and will use the following scenario to do so: Joe runs $\frac{3}{4}$ of a mile every 9 minutes.

Notice that there are two things to measure in this problem, namely time (in minutes) and distance (in miles). Therefore we can represent each of those quantities on its own number line. The key being that we know every 9 minutes corresponds to $\frac{3}{4}$ of a mile, so the time line at 9 minutes should match up with the distance line at $\frac{3}{4}$.

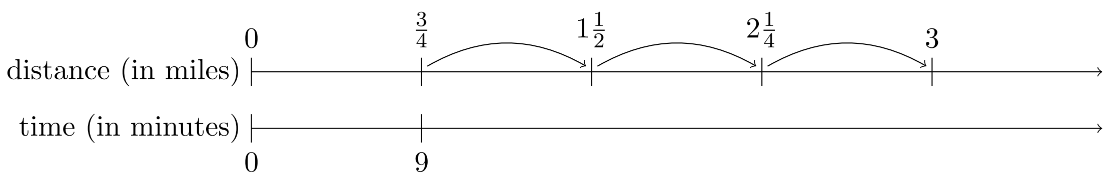


From this picture, we can then determine how far Joe runs in 18 minutes because a jump of 9 more minutes will correspond to a jump of $\frac{3}{4}$ of a mile.

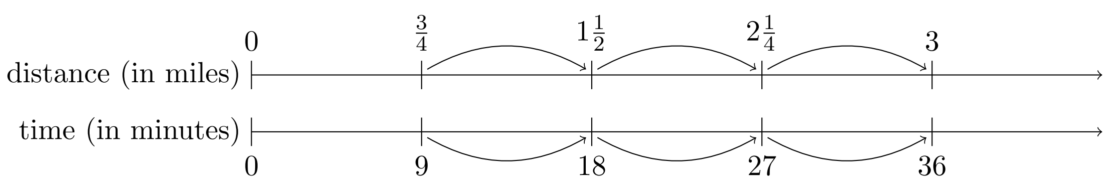


Therefore Joe can run a mile and a half in 18 minutes.

Let's now use the double number line to determine how long it will take Joe to run 3 miles. We will work on the distance line to see how many jumps of $\frac{3}{4}$ we need to get to 3 miles.



We can now make those same corresponding jumps on the time line.



Therefore it will take Joe 36 minutes to run 3 miles.

Question 12.4.1. How could we use the double number line to determine how long it will take Joe to run 1 mile?

Question 12.4.2. How could we use the double number line to determine how far Joe will run in 6 minutes?

The second tool that helps student solve proportional reasoning problems is ratio tables. Consider the following scenario.

A printer can print 15 pages in four minutes.

A ratio table is just a table whose entries are reasoned from one original entry. In this case the two quantities that could change are the number of pages and the time (in minutes). The initial table is shown below.

Number of pages	15	
Time (in minutes)	4	

Now if we let the printer work twice as long (so 8 minutes), then the printer will be able to print twice as many pages (so 30 pages).

Number of pages	15	30
Time (in minutes)	4	8

Notice that we use the context of the scenario to reason out new entries in our ratio table. Suppose we want to know how long it will take to print 10 pages.

Number of pages	15	30	10
Time (in minutes)	4	8	?

Looking at the previous entry (30 pages), we see that to get 10 pages we would be printing a third of the pages. Therefore that should take a third of the time the 30 pages took. One third of 8 is $\frac{8}{3}$.

Number of pages	15	30	10
Time (in minutes)	4	8	$\frac{8}{3}$

Since $\frac{8}{3} = 2\frac{2}{3}$, then 10 pages will take $2\frac{2}{3}$ minutes to print. Notice that the arithmetic we did was to just multiply both entries from column 2 by $\frac{1}{3}$ to get the new entries in column 3. But we found that arithmetic by reasoning within the context of the problem. In this way, we can see that multiplying both entries in any column by the same number will give a valid new entry in the table.

Question 12.4.3. If we added 2 to each entry in column 3, would that produce a valid entry in the ratio table?

Question 12.4.4. If we added column 2 to column 3, would that produce a valid entry in the ratio table?

Let's now consider the following scenario.

> A packaging company has 6 packaging machines. When all of the machines are running it takes 5 hours to package the daily shipment of batteries.

Suppose the machines do not always run everyday, so the number of machines may change which will affect the amount of time it takes. Therefore our table for this scenario is shown below.

Number of machines	6
Time (in hours)	5

Question 12.4.5. If we try to find new entries in the table, how is this scenario different than the previous scenario?

Question 12.4.6. How long would it take to package the batteries if only 2 of the machines were working?

As you will see in the exercises, ratio tables can be used to solve some rather difficult problems. In fact, you may find that some problems that you would normally think you would need algebra to do end up being easier if you use a ratio table. The strength of the ratio table is that students use their reasoning about the context to create new entries in the table, rather than just try to figure out what arithmetic to do out of context.

12.5 Exercises

1. Solve each of the following problems using a bar diagram, briefly explaining your steps along the way.

   (a) Sonia, Jeff, and Elena collect trading cards. Jeff has 3 more than Sonia, and Elena has twice as many as Sonia. Together they have 155 cards. How many cards does Jeff have?

   (b) Ron, Sonny, and Tom share a house, and they all do some of the work around the house. One week Ron put in 3 more hours than Sonny, and Tom put in half as much time as Sonny. Together they did 17 hours of work. How much time did they each spend on work around the house?

   (c) Fred and Joe are each putting some money toward an item that costs $253. Before buying the item, the amount of money that Fred has is four-fifths of the amount that Joe has. Joe then puts in half of his money and Fred puts in three-fourths of his to buy the item. How much money does each person have left after buying the item?

   (d) At Sac State, the number of graduating math majors is about two-fifths that of graduating liberal studies majors. Half of those liberal studies majors and three-fourths of those math majors went into the Sac State credential program, for a total of 176 students entering the program from those majors. How many graduating math majors and liberal studies majors were there?

2. Use a tape diagram to solve the following problems. Please briefly explain your steps along the way.

   (a) Max makes lemonade by mixing 2 cups of lemon juice for every 5 cups of water. If Max wants to make a total of 10 cups of lemonade (i.e. 10 cups total of lemon juice and water), how much lemon juice and water will he need?

   (b) The guitar factory produces 5 new guitars every 2 hours. How many hours would it take them to produce 17 guitars?

   (c) The ratio of Carly's marbles to Darcy's marbles is 5 to 3. After Carly gives a third of her marbles to another friend, Carly has 20 more marbles than Darcy. How many marbles does Darcy have?

3. Dr. Elce's "perfect purple" is made by mixing red and blue in a ratio of 2 to 3. Max is trying to make "perfect purple". He mixed 5 cups of red and 7 cups of blue. Use a tape diagram to help you answer the following questions.

   (a) Max's mixture is not "perfect purple". Does he need to add more red or more blue. Be sure to show how you arrived at your answer.

   (b) How much more of that color needs to be added to end up with "perfect purple" paint. Be sure to show how you arrived at your answer.

4. Max and Macy are making lemonade. Max mixes lemon juice and water in a ratio of 2 to 5. Macy mixes lemon juice and water in a ratio of 4 to 7.

   (a) If Max uses one cup of water, how much lemon juice will Max need?

   (b) What does the fraction $\frac{7}{4}$ have to do within the context of Macy's lemonade? Be sure you are talking about the **fraction seven fourths** and not the ratio 7 to 4.

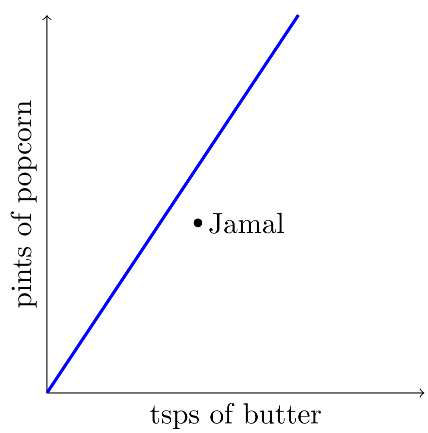
   (c) Whose lemonade is more "lemony". Be sure to justify your response in the context of the problem.

5. Theresa likes butter on her popcorn. She has determined that if she makes 3 pints of popcorn, she really wants exactly 2 teaspoons of butter on it.

   (a) She wants to make 8 pints of popcorn and wonders how much butter she'll need. Find the answer using a tape diagram.

   (b) One time she made 5 pints of popcorn and used 3 teaspoons of butter. Did this come out too buttery for her, not buttery enough, or just right? Answer this using a tape diagram.

   (c) If we plotted all of the possible amounts of popcorn and butter that would be acceptable to Theresa, we would get a line. That line is shown in blue below. Jamal is making buttered popcorn, and his combination of popcorn and butter is indicated by the point below. Is Jamal's recipe for buttered popcorn going to be too buttery for Theresa or not buttery enough? Justify your response.



6. Chris likes to add 4 tablespoons of chocolate mix to 12 ounces of milk.

   (a) Set up a ratio table and use it to determine how much chocolate mix Chris would add to 15 ounces of milk in order to get just the right flavor for his tastes. Use as many entries in the table as you wish, and explain how you found each new pair of entries.

   (b) Rita always adds 3 tablespoons of chocolate mix to 8 ounces of milk to make her chocolate milk. Whose chocolate milk would have a more chocolate taste, Chris's or Rita's? Explain how you know.

7. When Tina was in the Navy she loved the brownies and got the recipe. That recipe uses 72 cups of sugar, 36 cups of flour, and 24 cups of cocoa. Use a ratio table or a tape diagram to answer the following. Please make it clear how you arrived at your answer.

   (a) How much cocoa and sugar should she use if she uses 5 cups of flour?

   (b) How much of each ingredient should she use if she uses a total of 11 cups of sugar, flour, and cocoa combined?

8. Joanna bought some bulk cereal at the store and from her receipt she saw that she got 2.6 lbs and spent \$3.25.

   (a) She did a little work in a table of equivalent ratios (below) to figure out how much cereal she could buy.

cereal	2.6	0.8	3.4
dollars	3.25	1	4.25

   Explain what she computed in each step and why it is a valid entry in the table.

   (b) She decides to check her work by seeing how much it would cost to buy 3.4 lbs of cereal, so she makes another table of equivalent ratios.

cereal	2.6	1	3.4
dollars	3.25	1.25	4.25

   Explain what she computed in each step and why it is a valid entry in the table.

9. A party planner says that 3 pizzas will serve 5 people. Max wrote down the following ratio table.

# of pizzas	3	9	12	14
# of people	5	15	20	22

   (a) To get the second column in the ratio table Max multiplied both entries in the first column by 3. In the context of the problem, explain why this is a legal maneuver in the ratio table.

   (b) To get the third column in the ratio table Max added the entries in columns 1 and 2. In the context of the problem, explain why this is a legal maneuver in the ratio table.

   (c) To get the fourth column in the ratio table Max added 2 to both entries in column three. In the context of the problem, explain why this is **not** a legal maneuver in the ratio table.

10. Four pounds of Jelly Bellies cost \$5.

(a) Use ratio tables to determine the price per pound. Please show all work.

(b) Use ratio tables to determine the pounds per dollar. Please show all work.

(c) Janice has \$7 to spend on Jelly Bellies, and she wants to know how many pounds of Jelly Bellies she could buy. To answer her question, would it be easier to use the unit rate you found in part (a) or the unit rate you found in part (b)? Why?

(d) How many pounds can Janice buy?

11. RoboSmashers are robots that smash cans. It takes 3 RoboSmashers 5 minutes to smash 8 pounds of cans. Max set up the following ratio table.

# of RoboSmashers	3	9
minutes	5	15

(a) To get the second column Max multiplied both entries by 3. In the context of the problem, explain why that is not a legal move.

(b) How long will it take for 9 RoboSmashers to smash 8 pounds of cans? Explain how you arrived at your answer using the context of the problem.

12. Use a ratio table to solve the following problem. For each new column you create, write a sentence **in the context of the problem** that justifies the arithmetic you did to arrive at that new column.

If 8 wood-choppers can chop 9 cords of wood in 6.5 hours, how long will it take 4 wood-choppers to chop 3 cords, assuming that all wood-choppers work at the same rate?

13. A truck is traveling at a constant speed and goes 80 miles in 70 minutes. Use a double number line, briefly explaining your steps, to answer the following questions.

(a) How long will it take the truck to travel 100 miles?

(b) How far will the truck travel in 100 minutes?

(c) What is the trucks speed in miles per hour?

14. Janille went for a 45 minute run. Upon return she checked her phone, and her mileage was 7 miles. For parts 14a and 14b use a double line, briefly explaining your steps.

(a) What was Janille's average speed in miles per hour?

(b) What was Janille's average pace in minutes per mile?

(c) Which of the rates you found above is helpful in answering the following question? Why?

How long will it take Janille to run 5 miles?

15. On a map the distance between City A and City B is 2 inches. In the real word the distance between the two cities is 175 miles. Use a ratio table or double number line to answer the following questions. Briefly explain your steps.

(a) On the same map the distance between City A and City C is 3 inches. How far apart are they in the real world?

(b) In the real world City A and City D are 375 miles apart. How far apart will they be on the map?

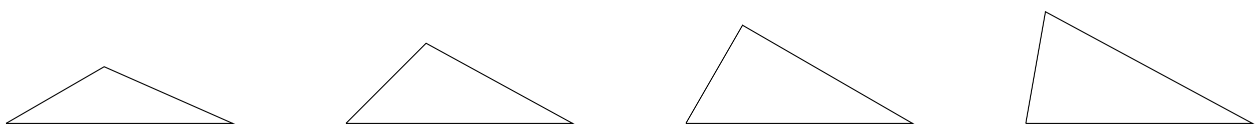
13 Geometry (Part II)

13.1 Pythagorean Theorem and Beyond

One of the most famous theorems in all of mathematics is the Pythagorean Theorem. There are hundreds and hundreds of proofs of this theorem. Even President Garfield came up with a proof of this theorem, some of which will be explored in the exercises. Although attributed to Pythagoras around 500 bce, there is evidence that it was actually known a thousand years before that. So needless to say it is an extremely old theorem.

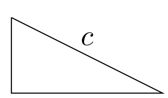
Theorem 13.1.1 (Pythagorean Theorem)**.** *If c is the length of the hypotenuse in a right triangle and a and b are the lengths of the legs, then $a^2 + b^2 = c^2$.*

Suppose I told you that I am thinking of a triangle and the two shortest sides have length 1 and 2. Could you guess what my triangle looked like? My guess is no, because there are infinitely many possibilities. Here are just a few.



On the other hand, what if I told you I am thinking of a right triangle and the two shortest sides have length 1 and 2. Could you now guess what my triangle looked like? Well, first we have to recognize that the longest side in a right triangle is the hypotenuse. So that means the right angle must be between the sides of length 1 and 2, as shown below.

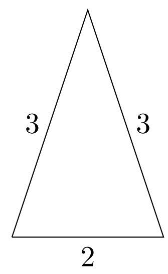
Once we figure that out, there is only one way to make this into a triangle. Therefore we know exactly what the triangle must look like just given those two pieces of information. In addition, because of the Pythagorean Theorem we also can figure out how long that side is that we need to draw in. That side is the hypotenuse, so let's call that length c. We can then use the Pythagorean Theorem to figure out what c is.



$$\begin{aligned} 1^2 + 2^2 &= c^2 \\ 5 &= c^2 \\ \sqrt{5} &= c \end{aligned}$$

Therefore all right triangles with legs of length 1 and 2 will have a hypotenuse of length $\sqrt{5}$.

Although, we certainly can solve direct right triangle problems like the one above, the Pythagorean Theorem has even farther reaching territory and can be used to solve problems that don't at first have anything to do with right triangles. For example, consider the isosceles triangle shown below.

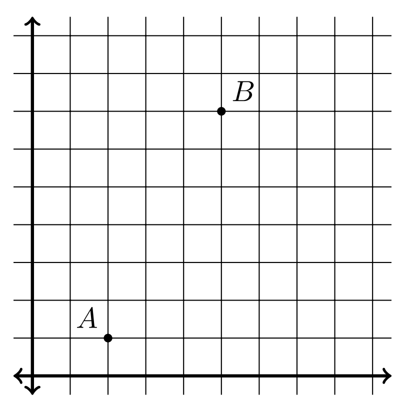


Clearly this is not a right triangle, but there are some hidden right triangles that will help you answer the following question.

Question 13.1.2. What is the area of the triangle above?

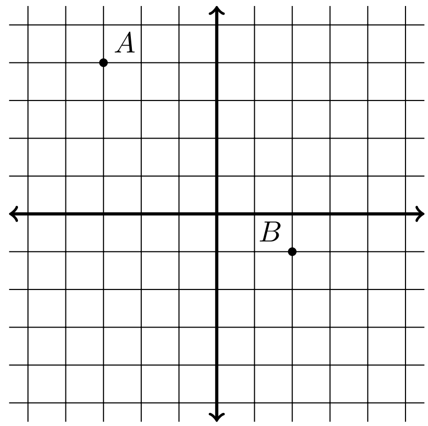
Let's continue to explore hidden right triangle problems. For our next example we want to work on a set of coordinate axes and consider two points. Suppose we want to find the distance between those two points. You probably remember learning at some point the distance formula. It is a formula that is often forgotten and misused by students because it has a rather complicated form. The distance formula says that the distance between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$. What is up with that? No wonder students don't remember this. Where on earth did this come from? Oh, if only students knew how simple it could be. Alas, distance problems are really just hidden right triangle problems.

Consider the points shown on the coordinate plane below.



Question 13.1.3. How can we use the Pythagorean Theorem to find the distance between points A and B.

Carefully analyzing our answer to the question above, we should be able to see the role $(x_2 - x_1)$ and $(y_2 - y_1)$ are playing in the distance formula, and also why there is a square root. Notice though in our example both points are in the first quadrant. Will this method still work if we have points in other quadrants? For example, let's consider the points shown below.



Question 13.1.4. Does the method you developed above work in this setting?

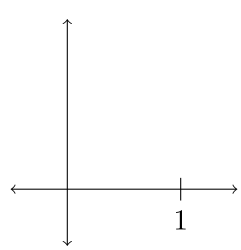
Question 13.1.5. Does $(x_2 - x_1)$ and $(y_2 - y_1)$ still make sense in this setting?

Notice that with enough examples like this students can come up with the distance formula. Yet another spot in mathematics where the students don't need to be told because they can figure it out! If students haven't found distance in a while they may forget the formula, however the added bonus here is that they know where it came from so if they figure what the formula says they can jot down a couple points, make a hidden right triangle, and remind themselves of the formula. Or if they've gone that far and they just need the distance they can just compute the distance without needing to write down the formula.

Now that we can find the distance between any two points, this gives us the ability to mark spots on the number line that we couldn't previously mark. Using the definition of a fraction we are able to mark any rational number on the number line, but what about the number $\sqrt{2}$? This is not a rational number, so how could we find where it is on the number line? Now, of course we could get a decimal approximation of $\sqrt{2}$ and mark that, and this would certainly get us a fairly accurate placement of $\sqrt{2}$ which would suffice. So really what we are asking here is can we place the number $\sqrt{2}$ without having to make an estimate? The answer is yes (probably not surprised by that) and Pythagorean Theorem is a crucial tool in this process. Before focusing on the number line, let's stick with working in two dimensions first.

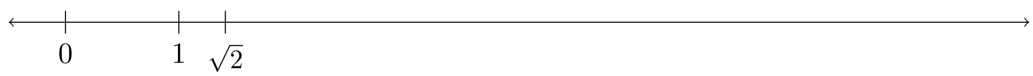
In order to avoid estimation as much as possible we are going to use the most simplistic tools possible to create points in specific locations and segments of given lengths. The two tools you are allowed to use are a compass and a straightedge. (Note that the straight edge does not have any marks on it and you are not allowed to make any marks on the straightedge.)

Question 13.1.6. Using only a straightedge and compass, can you find two points on the coordinate plane below so that the distance between them is $\sqrt{2}$?



Question 13.1.7. How can you use your work above to mark where $\sqrt{2}$ is on the x-axis?

Therefore we see that in order to mark $\sqrt{2}$ on a number line we can first go to two dimensions and find the distance, then transfer that distance to the number line. Once we know where $\sqrt{2}$ is on the number line, there are plenty of other numbers we can mark without having to use another application of the Pythagorean Theorem. The square root of two is marked on the number line below. In answering the following questions you should be able to work entirely on the number line without needing to go to two dimensions.



Question 13.1.8. Where is $3\sqrt{2}$ on the number line above?

Question 13.1.9. Where is $\sqrt{50}$ on the number line above?

We just chose $\sqrt{2}$ as an example to work with. However, the process we developed will work with lots of irrational numbers, and you will get a chance to play with this method in the exercises.

13.2 Angles

This section will discuss angles. It will also include a discussion of the relationship between angles and parallel lines.

13.3 Exercises

1. Problems involving different proofs of the pythagorean theorem will be added here.

2. Let C be the point with coordinates $(1, 2)$. In answering each of the questions please make it clear how you arrived at your answer.

   (a) If A has coordinates $(7, 10)$, how far is point A from point C?

   (b) Find two other points in the first quadrant so that the distance from those points to C is the same as the distance from A to C.

   (c) The point B is 10 units from C. If B lies in the first quadrant and has x-coordinate 3, what must its y-coordinate be?

   (d) The point with coordinates (x, y) is 10 units from C. Find an equation relating x and y.

   (e) In answering Problem 2d, you have discovered a "known" equation that is taught in high-school. What is it and how is it connected to what you have done above?

3. Make a copy of the number line in Appendix A and use a straightedge and compass to mark the following numbers on the number line. For each number, briefly describe the steps you took to place the number. For parts 3b - 3e, you may use your mark found in part 3a, but you may not construct any new right triangles.

   (a) $\sqrt{10}$
   (b) $3 + \sqrt{10}$
   (c) $2\sqrt{10}$
   (d) $\sqrt{90}$
   (e) $4 - \sqrt{10}$

4. Make a copy of the number line in Appendix A and use a straightedge and compass to mark the following numbers on the number line. For each number, briefly describe the steps you took to place the number.

   (a) $\sqrt{2}$
   (b) $\sqrt{3}$

5. Make a copy of the number line in Appendix A and use a straightedge and compass to mark the following numbers on the number line. For each number, briefly describe the steps you took to place the number.

   (a) $\sqrt{5}$
   (b) $\sqrt{21}$
   (c) $\sqrt{22}$

6. Consider the points $A = (2, 1)$ and $B = (7, 3)$.

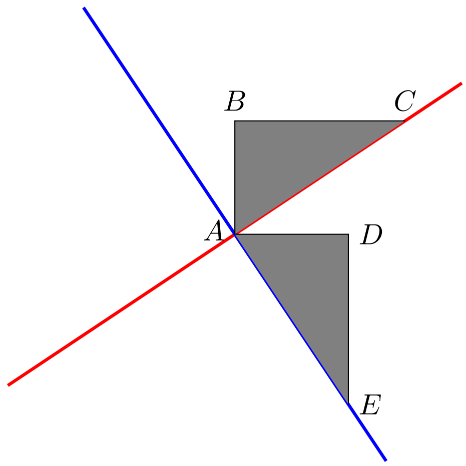
   (a) Explain how to use the Pythagorean Theorem to find the distance between these two points.

   (b) Suppose you want to find a point that is equidistant from A and B. Explain how you can use circles to find such a point.

   (c) Find at least four other points that are equidistant from A and B. What do you notice about the points?

   (d) In light of what you have done above, develop a method that allows you to find all the points equidistant from A and B using a compass and straight edge.

7. The red line and the blue line below are perpendicular.



(a) Explain how you know the two shaded triangles are similar.

(b) If the slope of the red line is $\frac{2}{3}$ and the length of segment AD is 2, what is the length of the segment DE?

(c) If the slope of the red line is $\frac{4}{7}$ and the length of segment AD is 4, what is the length of the segment DE?

(d) Analyzing your work above, what relationship have you discovered between the slopes of perpendicular lines. Explain.

8. Recall that if there are two points on the coordinate plane with coordinates (a, b) and (c, d), then the midpoint of the segment between those two points has coordinates $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$. Rather than trying to find the point halfway between two points though, they are trying to find a point a third of a way between the two points.

(a) Max says that the point $\left(\frac{a+c}{3}, \frac{b+d}{3}\right)$ will be a third of the way between the two points. Is Max correct? If so, explain how you know. If not, provide a counterexample to show that Max is not correct.

(b) Macy says that to find the point a third of the way you need to do the following:

> Draw a right triangle so that the segment between the points is the hypotenuse. Mark a third of the way on the horizontal leg and a third of the way on the vertical leg. Those two spots will give you the x- and y-coordinates for the point on the segment that is a third of the way.

Perform Macy's instructions using the points $(1, 2)$ and $(4, 3)$. Is the point you found a third of the way between? Give enough explanation so that it is clear how you arrived at your answer.

(c) Is Macy's method a valid way to find a point a third of the way between? Justify your response.

(d) According to Macy's method, what would a formula for finding a point a third of a way between (a, b) and (c, d) look like?